

The Rank-Width of Edge-Colored Graphs

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Abstract

Clique-width is a complexity measure of directed as well as undirected graphs. *Rank-width* is an equivalent complexity measure for undirected graphs and has good algorithmic and structural properties. It is in particular related to the *vertex-minor* relation. We discuss an extension of the notion of rank-width to edge-colored graphs. A C -colored graph is a graph where the arcs are colored with colors from the set C . There is not a natural notion of rank-width for C -colored graphs. We define two notions of rank-width for them, both based on a coding of C -colored graphs by edge-colored graphs where each edge has exactly one color from a field \mathbb{F} and named respectively \mathbb{F} -rank-width and \mathbb{F} -bi-rank-width. The two notions are equivalent to clique-width. We then present a notion of *vertex-minor* for \mathbb{F} -colored graphs and prove that \mathbb{F} -colored graphs of bounded \mathbb{F} -rank-width are characterised by a finite list of \mathbb{F} -colored graphs to exclude as vertex-minors. A cubic-time algorithm to decide whether a \mathbb{F} -colored graph has \mathbb{F} -rank-width (resp. \mathbb{F} -bi-rank-width) at most k , for fixed k , is also given. Graph operations to check MSOL-definable properties on \mathbb{F} -colored graphs of bounded rank-width are presented. A specialisation of all these notions to (directed) graphs without edge colors is presented, which shows that our results generalise the ones in undirected graphs.

Key words: rank-width; clique-width; local complementation; vertex-minor; excluded configuration; 2-structure; sigma-symmetry.

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1 Introduction

Clique-width [5,12] is a complexity measure for edge-colored graphs, *i.e.*, graphs where edges are colored with colors from a finite set. Clique-Width is more general than *tree-width* [33] because every class of graphs of bounded tree-width has bounded clique-width and the converse is false (complete undirected graphs have clique-width 2 and unbounded tree-width) [12]. Clique-width is an interesting complexity measure in algorithmic design. In fact every property expressible in *monadic second-order logic* (MSOL for short) can be checked in linear-time, provided the clique-width expression is given, on every graph that has small clique-width [11]. This result is important in complexity theory because many NP-complete problems are MS-definable properties, *e.g.*, 3-colorability. However, it is NP-complete to check if a graph has clique-width at most k when k is part of the input [17]. It is still open whether this problem is polynomial for fixed $k \geq 4$.

In their investigations of a recognition algorithm for undirected graphs of clique-width at most k , for fixed k , Oum and Seymour [31] introduced the notion of *rank-decomposition* and associated complexity measure *rank-width*, of undirected graphs. Rank-width is defined in a combinatorial way and is equivalent to the clique-width of undirected graphs in the sense that a class of graphs has bounded clique-width if and only if it has bounded rank-width [31]. But, being defined in a combinatorial way provides to rank-width better algorithmic properties than clique-width, in particular:

- for fixed k , there exists a cubic-time algorithm that decides whether the rank-width of an undirected graph is at most k and if so, constructs a rank-decomposition of width at most k [22];
- there exists a dual notion to rank-width, the notion of *tangle* [28,34]. This dual notion is interesting for getting certificates in recognition algorithms.

Since clique-width and rank-width of undirected graphs are equivalent, one way to check MSOL properties in undirected graphs of small rank-width is to transform a rank-decomposition into a clique-width expression [31]. However, an alternative characterization of rank-width in terms of graph operations has been proposed in [9]. It is thus possible to solve MSOL properties in graphs of small rank-width by using directly the rank-decomposition. This later result is important in a practical point of view because it avoids the exponent, that cannot be avoided [4,32], when transforming a rank-decomposition into a clique-width expression.

Another advantage of rank-width over clique-width is that it is invariant with respect to the *vertex-minor* relation (no such notion, except for induced sub-graph relation, is known for clique-width), *i.e.*, if H is a vertex-minor of G ,

then the rank-width of H is at most the rank-width of G [29]. Moreover, every class of undirected graphs of bounded rank-width is characterised by a finite list of undirected graphs to exclude as vertex-minors [29]. This later result generalises the one of Robertson and Seymour on undirected graphs of bounded tree-width [33].

Despite all these positive results of rank-width, the fact that clique-width is defined for graphs - directed or not, with edge colors or not - is an undeniable advantage over rank-width. It is thus natural to ask for a notion of rank-width for edge-colored graphs or at least for directed graphs without edge colors. Courcelle and Oum suggested in [13] a definition of rank-width for directed graphs as follows: Courcelle [6] described a graph transformation B from (directed) graphs to undirected bipartite graphs so that $f_1(\text{cwd}(B(G))) \leq \text{cwd}(G) \leq f_2(\text{cwd}(B(G)))$, for some functions f_1 and f_2 ; the rank-width of a (directed) graph is defined as the rank-width of $B(G)$. This definition can be extended to edge-colored graphs by using a similar coding (see [7, Chapter 6]). This definition gives a cubic-time algorithm that approximates the clique-width of edge-colored graphs. Another consequence is the proof of a weak version of the Seese's conjecture for edge-colored graphs [13]. However, this definition suffers from the following drawback: a vertex-minor of $B(G)$ does not always correspond to a coding of an edge-colored graph and similarly for the notion of *pivot-minor* (see for instance [20,29] for the definition of pivot-minor of undirected graphs).

We investigate in this paper a better notion of rank-width for edge-colored graphs. However, there is no unique natural way to extend rank-width to edge-colored graphs. We are looking for a notion that extends the one on undirected graphs and that can be used for directed graphs without edge colors. For that purposes, we will define the notion of *sigma-symmetric* matrices, which generalizes the notion of symmetric and skew-symmetric matrices. We then use this notion to represent edge-colored graphs by matrices over finite fields and derive, from this representation, a notion of rank-width, called \mathbb{F} -rank-width, that generalises the one of undirected graphs. We also define another notion of rank-width, called \mathbb{F} -bi-rank-width. We prove that the two parameters are equivalent to clique-width.

We then define a notion of vertex-minor for edge-colored graphs that extends the one on undirected graphs. We prove that \mathbb{F} -rank-width and \mathbb{F} -bi-rank-width are invariant with respect to this vertex-minor relation. We give a characterisation of edge-colored graphs of bounded \mathbb{F} -rank-width by excluded configurations. This result generalises the one on undirected graphs [29]. A generalisation of the pivot-minor relation is also presented.

The cubic-time recognition algorithm by Hliněný and Oum [22] can be adapted to give for fixed k , a cubic-time algorithm that decides if a given edge-colored

graph has \mathbb{F} -rank-width (resp. \mathbb{F} -bi-rank-width) at most k and if so, outputs an optimal rank-decomposition.

The two notions of rank-width of edge-colored graphs are specialised to directed graphs without colors on edges. All the results specialised to them.

The paper is organized as follows. In Section 2 we give some preliminary definitions and results. We recall in particular the definition of rank-width of undirected graphs. The first notion of rank-width of edge-colored graphs, called \mathbb{F} -rank-width, is studied in Section 3. We will define the notion of vertex-minor and pivot-minor, and prove that edge-colored graphs of bounded \mathbb{F} -rank-width are characterised by a finite list of edge-colored graphs to exclude as vertex-minors (resp. pivot-minors). A cubic-time recognition algorithm and a specialisation to directed graphs are also presented. We define our second notion of rank-width for edge-colored graphs called \mathbb{F} -bi-rank-width in Section 4. We also specialise it to directed graphs. In Section 5 we introduce some algebraic graph operations that generalise the ones in [9]. These operations will be used to characterise exactly the two notions of rank-width. They can be seen as alternatives to clique-width operations for solving MSOL properties. We conclude by some remarks and open questions in Section 6.

This paper is related to a companion paper where the authors introduce a decomposition of edge-colored graphs on a fixed field [24]. This decomposition plays a role similar to the *split decomposition* for the rank-width of undirected graphs. Particularly we show that the rank width of an edge-colored graph is exactly the maximum over the rank-width over all edge-colored prime graphs in the decomposition, and we give different characterisations of edge-colored graphs of rank-width one.

2 Preliminaries

For two sets A and B , we let $A \setminus B$ be the set $\{x \in A \mid x \notin B\}$. The power-set of a set V is denoted by 2^V . We often write x to denote the set $\{x\}$. The set of natural integers is denoted by \mathbb{N} .

We denote by $+$ and \cdot the binary operations of any field and by 0 and 1 the neutral elements of $+$ and \cdot respectively. For every prime number p and every positive integer k , we denote by \mathbb{F}_{p^k} the finite field of characteristic p and of order p^k . We recall that they are the only finite fields. We refer to [27] for our field terminology.

For sets R and C , an (R, C) -matrix is a matrix where the rows are indexed by elements in R and columns indexed by elements in C . For an (R, C) -matrix

M , if $X \subseteq R$ and $Y \subseteq C$, we let $M[X, Y]$ be the sub-matrix of M where the rows and the columns are indexed by X and Y respectively. We let rk be the matrix rank-function (the field will be clear from the context). We denote by M^T the transpose of a matrix M . The order of an (R, C) -matrix is defined as $|R| \times |C|$. We often write $k \times \ell$ -matrix to denote a matrix of order $k \times \ell$. For positive integers k and ℓ , we let $O_{k,\ell}$ be the null $k \times \ell$ -matrix and I_k the identity $k \times k$ -matrix, or respectively O and I when the size is clear in the context.

We use the standard graph terminology, see for instance [15]. A graph G is a couple (V_G, E_G) where V_G is the set of vertices and $E_G \subseteq V_G \times V_G$ is the set of edges. A graph G is said to be *oriented* if $(x, y) \in E_G$ implies $(y, x) \notin E_G$, and it is said *undirected* if $(x, y) \in E_G$ implies $(y, x) \in E_G$. An edge between x and y in an undirected graph is denoted by xy (equivalently yx). For a graph G , we denote by $G[X]$, called the sub-graph of G induced by $X \subseteq V_G$, the graph $(X, E_G \cap (X \times X))$; we let $G-X$ be the sub-graph $G[V_G \setminus X]$. The degree of a vertex x in an undirected graph G is the cardinal of the set $\{y \mid xy \in E_G\}$. Two graphs G and H are *isomorphic* if there exists a bijection $h : V_G \rightarrow V_H$ such that $(x, y) \in E_G$ if and only if $(h(x), h(y)) \in E_H$. We call h an *isomorphism* between G and H . All graphs are finite and loop-free (*i.e.* for every $x \in V_G$, $(x, x) \notin E_G$).

A *tree* is an acyclic connected undirected graph. In order to avoid confusions in some lemmas, we will call *nodes* the vertices of trees. The nodes of degree 1 are called *leaves* and the set of leaves in a tree T is denoted by L_T . A *sub-cubic tree* is an undirected tree such that the degree of each node is at most 3. A tree T is *rooted* if it has a distinguished node r , called the *root* of T . For convenience, we will consider a rooted tree as an oriented graph such that the underlying graph is a tree, and such that all nodes are reachable from the root by a directed path. For a tree T and an edge e of T , we let $T-e$ denote the graph $(V_T, E_T \setminus e)$.

Let C be a (possibly infinite) set that we call the *colors*. A C -graph G is a tuple (V_G, E_G, ℓ_G) where (V_G, E_G) is a graph and $\ell_G : E_G \rightarrow C$ is a function. Its associated *underlying graph* $u(G)$ is the graph (V_G, E_G) . Two C -graph G and H are isomorphic if there is an isomorphism h between (V_G, E_G) and (V_H, E_H) such that for every $(x, y) \in E_G$, $\ell_G((x, y)) = \ell_H((h(x), h(y)))$. We call h an *isomorphism* between G and H . We let $\mathcal{G}(C)$ be the set of C -graphs for a fixed color set C . Even though we authorise infinite color sets in the definition, most of results in this article are valid only when the color set is finite. It is worth noticing that an edge-uncolored graph can be seen as an edge-colored graph where all the edges have the same color.

Remark 2.1 (Multiple colors per edge) *In our definition, an edge in a C -graph can only have one color. However, this is not restrictive because if in*

an edge-colored graph an edge can have several colors from a set C , we just extend C to 2^C .

Remark 2.2 (2-structures and edge-colored graphs) A 2-structure [16] is a pair (D, R) where D is a finite set and R is an equivalence relation on the set $D_2 = \{(x, y) \mid x, y \in D \text{ and } x \neq y\}$. Every 2-structure (D, R) can be seen as a C -colored graph $G = (D, D_2, \ell)$ where $C := \{[e] \mid [e] \text{ is an equivalence class of } R\}$ and for every edge e , $\ell(e) := [e]$. Equivalently, every C -graph G can be seen as a 2-structure (V_G, R) where eRe' if and only if $\ell_G(e) = \ell_G(e')$ and all the non-edges in G are equivalent with respect to R .

A *parameter* on $\mathcal{G}(C)$ is a function $wd : \mathcal{G}(C) \rightarrow \mathbb{N}$ that is invariant under isomorphism. Two parameters on $\mathcal{G}(C)$, say wd and wd' , are *equivalent* if there exist two integer functions f and g such that for every edge-colored graph $G \in \mathcal{G}(C)$, $f(wd'(G)) \leq wd(G) \leq g(wd'(G))$.

The *clique-width*, denoted by cwd , is a graph parameter defined by Courcelle et al. [5,12]. Most of the investigations concern edge-uncolored graphs. However, its edge-colored version has been investigated these last years (see [3,18]). Note that the clique-width is also defined in more general case where edges can have several colors.

We finish these preliminaries by the notion of terms. Let \mathcal{F} be a set of binary and unary function symbols and \mathcal{C} a set of constants. We denote by $T(\mathcal{F}, \mathcal{C})$ the set of finite well-formed terms built with $\mathcal{F} \cup \mathcal{C}$. Notice that the syntactic tree of a term is rooted.

A *context* is a term in $T(\mathcal{F}, \mathcal{C} \cup \{u\})$ having a single occurrence of the variable u (a nullary symbol). We denote by $Cxt(\mathcal{F}, \mathcal{C})$ the set of contexts. We denote by Id the particular context u . If s is a context and t a term, we let $s \bullet t$ be the term in $T(\mathcal{F}, \mathcal{C})$ obtained by substituting t for u in s .

2.1 Rank-Width and Vertex-Minor of Undirected Graphs

Despite the interesting algorithmic results [11], clique-width suffers from the lack of a recognition algorithm. In their investigations for a recognition algorithm, Oum and Seymour introduced the notion of *rank-width* [31], which approximates the clique-width of undirected graphs. Let us first define some notions.

Let V be a finite set and $f : 2^V \rightarrow \mathbb{N}$ a function. We say that f is *symmetric* if for any $X \subseteq V$, $f(X) = f(V \setminus X)$; f is *submodular* if for any $X, Y \subseteq V$, $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$.

A *layout* of a finite set V is a pair (T, \mathcal{L}) of a sub-cubic tree T and a bijective function $\mathcal{L} : V \rightarrow L_T$. For each edge e of T , the connected components of $T-e$ induce a bipartition $(X_e, V \setminus X_e)$ of L_T , and thus a bipartition $(X^e, V \setminus X^e) = (\mathcal{L}^{-1}(X_e), \mathcal{L}^{-1}(V \setminus X_e))$ of V (we will omit the sub or sup-script e when the context is clear).

Let $f : 2^V \rightarrow \mathbb{N}$ be a symmetric function and (T, \mathcal{L}) a layout of V . The *f-width of each edge e of T* is defined as $f(X^e)$ and the *f-width of (T, \mathcal{L})* is the maximum *f-width* over all edges of T . The *f-width of V* is the minimum *f-width* over all layouts of V .

Definition 2.3 (Rank-width of undirected graphs [29,31]) *For every undirected graph G , we let M_G be its adjacency (V_G, V_G) -matrix where $M_G[x, y] := 1$ if and only if $xy \in E_G$. For every graph G , we let $\text{cutrk}_G : 2^{V_G} \rightarrow \mathbb{N}$ where $\text{cutrk}_G(X) := \text{rk}(M_G[X, V_G \setminus X])$, where rk is the matrix rank over \mathbb{F}_2 . This function is symmetric. The rank-width of an undirected graph G , denoted $\text{rwd}(G)$, is the cutrk_G -width of V_G .*

Rank-Width has several structural and algorithmic results, see for instance [9,22,29]. In particular, for fixed k , there exists a cubic-time algorithm for recognizing undirected graphs of rank-width at most k [22]. Moreover, rank-width is related to a relation on undirected graphs, called *vertex-minor*.

Definition 2.4 (Local complementation, Vertex-minor [29]) *For an undirected graph G and a vertex x of G , the local complementation at x , denoted by $G * v$, consists in replacing the sub-graph induced on the neighbors of x by its complement. A graph H is a vertex-minor of a graph G if H can be obtained from G by applying a sequence of local complementations and deletions of vertices.*

Authors of [2,20,29] also introduced the *pivot* operation on an edge xy , denoted by $G \wedge xy = G * x * y * x = G * y * x * y$. An interesting theorem relating rank-width and the notion of vertex-minor is the following.

Theorem 2.5 ([29]) *For every positive integer k , there exists a finite list \mathcal{C}_k of undirected graphs such that an undirected graph has rank-width at most k if and only if it does not contain as vertex-minor any graph isomorphic to a graph in \mathcal{C}_k .*

In the next section, we define the notion of rank-width of edge-colored graphs and generalize Theorem 2.5 to them.

3 \mathbb{F} -Rank-Width of σ -Symmetric \mathbb{F}^* -Graphs

We want a notion of rank-width for edge-colored graphs that generalises the one on undirected graphs. For that purposes, we will identify each color by an non-zero element of a field. This representation will allow us to define the rank-width of edge-colored graphs by using rank matrices.

Let \mathbb{F} be a field, and let $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ (where 0 is the zero of \mathbb{F}). One can note that there is a natural bijection between the class of \mathbb{F}^* -graphs and the class of \mathbb{F} -graphs with complete underlying graph (replace every non-edge by an edge of color 0). From now on, we do not distinguish these two classes, and we let $\ell_G((x, y)) = 0$ for all $(x, y) \notin E_G$.

We can represent every \mathbb{F}^* -graph G by a (V_G, V_G) -matrix M_G such that $M_G[x, y] := \ell_G((x, y))$ for every $x, y \in V_G$ with $x \neq y$, and $M_G[x, x] := 0$ for every $x \in V_G$.

Let $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ be a bijection. We recall that σ is an *involution* if $\sigma(\sigma(a)) = a$ for all $a \in \mathbb{F}$. We call σ a *sesqui-morphism* if σ is an involution, and the mapping $[x \mapsto \sigma(x)/\sigma(1)]$ is an automorphism. It is worth noticing that if $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ is a sesqui-morphism, then $\sigma(0) = 0$ and for every $a, b \in \mathbb{F}$, $\sigma(a + b) = \sigma(a) + \sigma(b)$ (*i.e.* σ is an automorphism for the addition). Moreover, we have the following notable equalities.

Proposition 3.1 *If σ is a sesqui-morphism, then*

$$\begin{aligned}\sigma(a \cdot b) &= \frac{\sigma(a) \cdot \sigma(b)}{\sigma(1)}, \\ \sigma\left(\frac{a}{b}\right) &= \frac{\sigma(1) \cdot \sigma(a)}{\sigma(b)}, \\ \sigma\left(\frac{a \cdot b}{c}\right) &= \frac{\sigma(a) \cdot \sigma(b)}{\sigma(c)}.\end{aligned}$$

A \mathbb{F}^* -graph is σ -*symmetric* if the underlying graph is undirected, and for every arc (x, y) , $\ell_G((x, y)) = a$ if and only if $\ell_G((y, x)) = \sigma(a)$. Clearly, if G is a σ -symmetric \mathbb{F}^* -graph, then $M_G[x, y] = \sigma(M_G[y, x])$. We denote by $\mathcal{S}(\mathbb{F})$ (respectively $\mathcal{S}(\mathbb{F}, \sigma)$) the set of \mathbb{F}^* -graphs (respectively σ -symmetric \mathbb{F}^* -graphs). Note that $\mathcal{S}(\mathbb{F}) = \mathcal{G}(\mathbb{F}^*)$.

To represent a C -graph, one can take an injection from C to \mathbb{F}^* for a large enough field \mathbb{F} . Notice that the representation is not unique: on one hand, several incomparable fields are possible for \mathbb{F} , and on the other hand, the representation depends on the injection from C to \mathbb{F}^* . For example, oriented graphs can be represented by a \mathbb{F}_3^* -graph or by a \mathbb{F}_4^* -graph (see Section 3.4). Two different representations can give two different rank-width parameters,

but the two parameters are equivalent when C is finite (direct consequence of Proposition 3.11).

Let \mathbb{F} be a finite field of characteristic p and order q . We will prove that every \mathbb{F}^* -graph can be seen as a $\tilde{\sigma}$ -symmetric $(\mathbb{F}^2)^*$ -graph for some sesqui-morphism $\tilde{\sigma}$, where \mathbb{F}^2 is an algebraic extension of \mathbb{F} of order 2. Let us first make some observations.

Lemma 3.2 *There exists an element p in \mathbb{F}^* such that the polynomial $X^2 - p(X + 1)$ has no root in \mathbb{F} .*

Proof. There exist $|\mathbb{F}| - 1$ distinct polynomials of the form $X^2 - p(X + 1)$, $p \neq 0$. We first notice that 0 or -1 cannot be a root of $X^2 - p(X + 1)$, for any $p \in \mathbb{F}^*$. Now, two such polynomials cannot have a common root. Assume the contrary and let α be a root of $X^2 - p(X + 1)$ and of $X^2 - p'(X + 1)$ with $p \neq p'$. Then $(\alpha + 1) \cdot (p - p') = 0$, *i.e.* $p = p'$ since $\alpha \neq -1$, a contradiction. Since -1 and 0 cannot be the roots of any of the polynomials, we have at most $|\mathbb{F}| - 2$ possible roots. Therefore, there exists a p such that $X^2 - p(X + 1)$ has no root in \mathbb{F} . \square

We can now construct an algebraic extension of the finite field \mathbb{F} . Let $p \in \mathbb{F}^*$ such that $X^2 - p(X + 1)$ has no root in \mathbb{F} and let \mathbb{F}^2 be isomorphic to the field $\mathbb{F}[X] \bmod (X^2 - p(X + 1))$ (*i.e.* \mathbb{F}^2 is the finite field of characteristic p and order q^2). Let $\alpha := X \bmod (X^2 - p(X + 1))$. Then every element of \mathbb{F}^2 is a polynomial on α of the form $a_0 + a_1\alpha$ where $a_0, a_1 \in \mathbb{F}$. Moreover, α is a root of $X^2 - p(X + 1)$ in \mathbb{F}^2 .

We let $\gamma := 1 - p^{-1}\alpha$ and $\tau := p^{-1}\alpha$ be in \mathbb{F}^2 . Notice that $\alpha = p\tau$ and $1 = \gamma + \tau$.

Lemma 3.3 *We have the following equalities:*

$$\begin{aligned}\gamma^2 &= (1 + p^{-1})\gamma + p^{-1}\tau, \\ \tau^2 &= p^{-1}\gamma + (1 + p^{-1})\tau, \\ \gamma \cdot \tau &= p^{-1}\gamma + p^{-1}\tau.\end{aligned}$$

To every pair of elements in \mathbb{F} , we associate an element in \mathbb{F}^2 by letting $\tilde{f} : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}^2$ where, for every $(a, b) \in \mathbb{F} \times \mathbb{F}$, $\tilde{f}(a, b) := a\gamma + b\tau$.

Lemma 3.4 *\tilde{f} is a bijection.*

For the sesqui-morphism in \mathbb{F}^2 , we let $\tilde{\sigma} : \mathbb{F}^2 \rightarrow \mathbb{F}^2$ where $\tilde{\sigma}(a\gamma + b\tau) := b\gamma + a\tau$. One easily verifies that $\tilde{\sigma}(\tilde{\sigma}(\beta)) = \beta$ for all $\beta \in \mathbb{F}^2$.

Lemma 3.5 *$\tilde{\sigma}$ is an automorphism.*

Proof. An easy computation shows that $\tilde{\sigma}((a\gamma + b\tau) + (c\gamma + d\tau)) = \tilde{\sigma}(a\gamma + b\tau) + \tilde{\sigma}(c\gamma + d\tau)$. For the product, we have:

$$\begin{aligned}\tilde{\sigma}((a\gamma + b\tau) \cdot (c\gamma + d\tau)) &= \tilde{\sigma}(ac\gamma^2 + (ad + bc)\gamma\tau + bd\tau^2) \\ &= ac\tilde{\sigma}(\gamma^2) + (ad + bc)\tilde{\sigma}(\gamma\tau) + bd\tilde{\sigma}(\tau^2)\end{aligned}$$

and

$$\begin{aligned}\tilde{\sigma}(a\gamma + b\tau) \cdot \tilde{\sigma}(c\gamma + d\tau) &= (b\gamma + a\tau) \cdot (d\gamma + c\tau) \\ &= bd\gamma^2 + (ad + bc)\gamma\tau + ac\tau^2.\end{aligned}$$

By Lemma 3.3, $\tilde{\sigma}(\gamma^2) = \tau^2$, $\tilde{\sigma}(\tau^2) = \gamma^2$ and $\tilde{\sigma}(\gamma\tau) = \gamma\tau$. This concludes the proof of the lemma. \square

For every \mathbb{F}^* -graph G , we let \tilde{G} be the $(\mathbb{F}^2)^*$ -graph $(V_G, E_G, \ell_{\tilde{G}})$ where, for every two distinct vertices x and y ,

$$\ell_{\tilde{G}}((x, y)) := \tilde{f}(\ell_G((x, y)), \ell_G((y, x))).$$

By the definitions of \tilde{G} and $\tilde{\sigma}$, and Lemmas 3.3-3.5, we get the following.

Proposition 3.6 *The mapping $[G \mapsto \tilde{G}]$ from $\mathcal{S}(\mathbb{F})$ to $\mathcal{S}(\mathbb{F}^2, \tilde{\sigma})$ is a bijection and for every \mathbb{F}^* -graph G , \tilde{G} is $\tilde{\sigma}$ -symmetric. Moreover, for two \mathbb{F}^* -graphs G and H , \tilde{G} and \tilde{H} are isomorphic if and only if G and H are isomorphic.*

Nevertheless, two different mappings can give two different rank-width parameters. But again, since \mathbb{F} is finite, the parameters are equivalent.

If \mathbb{F} is infinite, a mapping from $\mathcal{S}(\mathbb{F})$ to $\mathcal{S}(\mathbb{G}, \sigma)$ is not always possible with the previous construction. For example, a mapping is possible from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{C}, \sigma)$ with $f(a, b) = (1+i)a + (1-i)b$ and $\sigma(a+ib) = a-ib$ (where $a, b \in \mathbb{R}$), but the construction fails for $\mathbb{F} = \mathbb{C}$ since the complexes are algebraically closed.

From now on, we will focus our attention to sigma-symmetric \mathbb{F}^* -graphs. In Section 3.1 we define the notion of \mathbb{F} -rank-width. The notion of *vertex-minor* for \mathbb{F}^* -graphs is presented in Section 3.2 and we prove that sigma-symmetric \mathbb{F}^* -graphs of \mathbb{F} -rank-width at most k are characterised by a finite list of sigma-symmetric \mathbb{F}^* -graphs to exclude as vertex-minors. We prove in Section 3.3 that \mathbb{F}^* -graphs of \mathbb{F} -rank-width at most k , for fixed k , can be recognised in cubic-time when \mathbb{F} is finite. A specialisation to graphs without colors on edges is presented in Section 3.4.

3.1 Rank-Width of σ -symmetric \mathbb{F}^* -Graphs

Along this section, we let \mathbb{F} be a fixed field (of characteristic p and of order q if \mathbb{F} is finite), and we let $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ be a fixed sesqui-morphism. We recall that if G is a \mathbb{F}^* -graph, we denote by M_G the (V_G, V_G) -matrix where:

$$M_G[x, y] := \begin{cases} \ell_G((x, y)) & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.7 (Cut-Rank Functions) *The \mathbb{F} -cut-rank function of a σ -symmetric \mathbb{F}^* -graph G is the function $\mathbb{F}\text{-cutrk}_G : 2^{V_G} \rightarrow \mathbb{N}$ where $\mathbb{F}\text{-cutrk}_G(X) = \text{rk}(M_G[X, V_G \setminus X])$ for all $X \subseteq V_G$.*

Lemma 3.8 *For every σ -symmetric \mathbb{F}^* -graph G , the function $\mathbb{F}\text{-cutrk}_G$ is symmetric and submodular.*

We first recall the *submodular inequality* of the matrix rank-function.

Proposition 3.9 [29, Proposition 4.1] *Let M be an (R, C) -matrix over a field \mathbb{F} . Then for all $X_1, Y_1 \subseteq R$ and $X_2, Y_2 \subseteq C$,*

$$\text{rk}(M[X_1, X_2]) + \text{rk}(M[Y_1, Y_2]) \geq \text{rk}(M[X_1 \cup Y_1, X_2 \cap Y_2]) + \text{rk}(M[X_1 \cap Y_1, X_2 \cup Y_2]).$$

Proof of Lemma 3.8. Let X and Y be subsets of V_G . We let $A_1 = M_G[X, V_G \setminus X]$ and $A_2 = M_G[Y, V_G \setminus Y]$. We first prove the first statement.

We let M' be the $(V_G \setminus X, X)$ -matrix where $M'[y, x] = \sigma(A_1[x, y])/\sigma(1)$. Since σ is a sesqui-morphism, the mapping $[x \mapsto \sigma(x)/\sigma(1)]$ is an automorphism and then $\text{rk}(M') = \text{rk}((A_1)^T) = \text{rk}(A_1)$. But, $M_G[V_G \setminus X, X] = \sigma(1) \cdot M'$. Then, $\text{rk}(M_G[V_G \setminus X, X]) = \text{rk}(M') = \text{rk}(M_G[X, V_G \setminus X])$.

For the second statement, we have by definition and Proposition 3.9,

$$\begin{aligned} \mathbb{F}\text{-cutrk}_G(X) + \mathbb{F}\text{-cutrk}_G(Y) &= \text{rk}(A_1) + \text{rk}(A_2) \\ &\geq \text{rk}(M_G[X \cup Y, V_G \setminus X \cap V_G \setminus Y]) + \text{rk}(M_G[X \cap Y, V_G \setminus X \cup V_G \setminus Y]). \end{aligned}$$

Since $V_G \setminus X \cap V_G \setminus Y = V_G \setminus (X \cup Y)$ and $V_G \setminus X \cup V_G \setminus Y = V_G \setminus (X \cap Y)$, the second statement holds. \square

Definition 3.10 (\mathbb{F} -rank-width) *The \mathbb{F} -rank-width of a σ -symmetric \mathbb{F}^* -graph G , denoted by $\mathbb{F}\text{-rwd}(G)$, is the $\mathbb{F}\text{-cutrk}_G$ -width of V_G .*

This definition generalises the one for undirected graphs. If we let σ_1 be the identity automorphism on \mathbb{F}_2 , every undirected graph is a σ_1 -symmetric \mathbb{F}_2^* -graph. Moreover, for every undirected graph G , the functions cutrk_G and

\mathbb{F}_2 -cutrk $_G$ are equal. It is then clear that the definition of rank-width given in Section 2.1 coincides with the one of \mathbb{F}_2 -rank-width.

One can easily verify that the \mathbb{F} -rank-width of a σ -symmetric \mathbb{F}^* -graph is the maximum of the \mathbb{F} -rank-width of its maximum connected components. The following proposition, which says that \mathbb{F} -rank-width and clique-width are equivalent when \mathbb{F} is finite, has an easy proof. We omit it because its proof is an easy adaptation of the one comparing rank-width and clique-width of undirected graphs [31, Proposition 6.3].

Proposition 3.11 *Let G be a σ -symmetric \mathbb{F}^* -graph. Then, $\mathbb{F}\text{-rwd}(G) \leq \text{cwd}(G) \leq 2 \cdot q^{\mathbb{F}\text{-rwd}(G)} - 1$.*

It is also easy to show that the clique-width and the \mathbb{F} -rank-width are equivalent if \mathbb{F} is infinite but C is finite. In [23,24], authors present a decomposition related to \mathbb{F} -rank-width and different characterizations of graphs of \mathbb{F} -rank-width 1.

3.2 Vertex-Minor and Pivot-Minor

Bouchet generalised in [2] the notion of local complementation to all graphs (undirected or not). We recall that a graph G is a \mathbb{F}_2^* -graph and then is represented by a (V_G, V_G) -matrix M_G over \mathbb{F}_2 where $M_G[x, y] := 1$ if and only if $(x, y) \in E_G$. A *local complementation at x* of G is the graph represented by the matrix M'_G over \mathbb{F}_2 where $M'_G[z, y] = M_G[z, y] + M_G[z, x] \cdot M_G[x, y]$. This definition coincides with the one on undirected graphs when G is undirected. We will extend it to \mathbb{F}^* -graphs. We say that λ in \mathbb{F}^* is σ -compatible if $\sigma(\lambda) = \lambda \cdot \sigma(1)^2$.

Definition 3.12 (λ -local complementation) *Let λ in \mathbb{F}^* . Let G be a \mathbb{F}^* -graph and x a vertex of G . The λ -local complementation at x of G is the \mathbb{F}^* -graph $G * (x, \lambda)$ represented by the (V_G, V_G) -matrix $M_{G*(x, \lambda)}$ where:*

$$M_{G*(x, \lambda)}[z, t] := \begin{cases} M_G[z, t] + \lambda \cdot M_G[z, x] \cdot M_G[x, t] & \text{if } x \notin \{z, t\}, \\ M_G[z, t] & \text{otherwise.} \end{cases}$$

One can easily verify that for every \mathbb{F}^* -graph G and every vertex x of G , the adjacency matrix of $G*(x, \lambda)$ is obtained by modifying the sub-matrix induced by the neighbors of x . Then for every vertex y of G , $M_G[x, y] = M_{G*(x, \lambda)}[x, y]$.

Definition 3.13 (locally equivalent, vertex-minor) *A \mathbb{F}^* -graph H is locally equivalent to a \mathbb{F}^* -graph G if H is obtained by applying a sequence of λ -local complementations to G with $\lambda \in \mathbb{F}^*$. We call H a vertex-minor of G if*

$H = G'[X]$ for some $X \subseteq V_G$ and G' is locally equivalent to G . Moreover, H is a proper vertex-minor of G if $X \subsetneq V_G$.

In this section, we are interested in σ -symmetric graphs, thus we have to restrict ourselves to a subset of local complementations which preserve the σ -symmetry. We now prove that λ -local complementation is well defined on σ -symmetric graphs when λ is σ -compatible.

Lemma 3.14 *Let G be a σ -symmetric \mathbb{F}^* -graph and let $\lambda \in \mathbb{F}^*$ be σ -compatible. Then every λ -local complementation of G is also σ -symmetric.*

Proof. Let $H := G * (x, \lambda)$ for some σ -compatible λ . It is sufficient to prove that $M_H[t, z] = \sigma(M_H[z, t])$ for any $z, t \in V_G$, $z \neq t$.

$$\begin{aligned}
M_H[t, z] &= M_G[t, z] + \lambda \cdot M_G[t, x] \cdot M_G[x, z] \\
&= \sigma(M_G[z, t]) + \lambda \cdot \sigma(M_G[x, t]) \cdot \sigma(M_G[z, x]) \\
&= \sigma(M_G[z, t]) + \lambda \cdot \sigma(1) \cdot \sigma(M_G[z, x] \cdot M_G[x, t]) \\
&= \sigma(M_G[z, t]) + \sigma(\lambda) \cdot \sigma^{-1}(1) \cdot \sigma(M_G[z, x] \cdot M_G[x, t]) \\
&= \sigma(M_G[z, t]) + \sigma(\lambda \cdot M_G[z, x] \cdot M_G[x, t]) \\
&= \sigma(M_G[z, t] + \lambda \cdot M_G[z, x] \cdot M_G[x, t]) \\
&= \sigma(M_H[z, t]).
\end{aligned}$$

Definition 3.15 (σ -locally-equivalent, σ -vertex-minor) *A \mathbb{F}^* -graph H is σ -locally-equivalent to a σ -symmetric \mathbb{F}^* -graph G if H is obtained by applying a sequence of λ -local-complementations to G with σ -compatibles λ . We call H a σ -vertex-minor of G if $H = G'[X]$ for some $X \subseteq V_G$ and G' is σ -locally-equivalent to G . Moreover, H is a proper σ -vertex-minor of G if $X \subsetneq V_G$.*

Note that if no σ -compatible $\lambda \in \mathbb{F}^*$ exists, H is a σ -vertex-minor of G if and only if H is an induced subgraph of G .

Remark 3.16 *Lemma 3.14 shows that λ -local complementation is well-defined on σ -symmetric \mathbb{F}^* -graphs for σ -compatible λ . Moreover, one can easily verify that when \mathbb{F} is the field \mathbb{F}_2 , this notion of 1-local complementation coincides with the one defined by Bouchet in [2].*

The following lemma proves that σ -local-complementation does not increase \mathbb{F} -rank-width.

Lemma 3.17 *Let G be a σ -symmetric \mathbb{F}^* -graph and x a vertex of G . For every subset X of V_G ,*

$$\mathbb{F}\text{-cutrk}_{G*(x, \lambda)}(X) = \mathbb{F}\text{-cutrk}_G(X).$$

Proof. We can assume that $x \in X$ since \mathbb{F} -cutrk $_G$ is a symmetric function (Lemma 3.8). For each $y \in X$, the σ -local-complementation at x results in adding a multiple of the row indexed by x to the row indexed by y . Precisely, we obtain $M_{G^{*(x,\lambda)}}[y, V_G \setminus X]$ by adding $\lambda \cdot M_G[y, x] \cdot M_G[x, V_G \setminus X]$ to $M_G[y, V_G \setminus X]$. This operation is repeated for all $y \in X$. In each case, the rank of the matrix does not change. Hence, \mathbb{F} -cutrk $_{G^{*(x,\lambda)}}(X) = \mathbb{F}$ -cutrk $_G(X)$. \square

Unfortunately, such a σ -compatible λ does not always exist. For instance, if the field is \mathbb{F}_3 and σ is such that $\sigma(x) = -x$ (see Section 3.4), no σ -compatible λ does exist. We present now an other \mathbb{F}^* -graph transformation which is defined for every couple (\mathbb{F}, σ) .

Definition 3.18 (Pivot-complementation) *Let G be a σ -symmetric \mathbb{F}^* -graph, and x and y two vertices of G such that $\ell_G((x, y)) \neq 0$. The pivot-complementation at xy of G is the \mathbb{F}^* -graph $G \wedge xy$ represented by the (V_G, V_G) -matrix $M_{G \wedge xy}$ where $M_{G \wedge xy}[z, z] := 0$ for every $z \in V_G$, and for every $z, t \in V_G \setminus \{x, y\}$ with $z \neq t$:*

$$M_{G \wedge xy}[z, t] := M_G[z, t] - \frac{M_G[z, x] \cdot M_G[y, t]}{M_G[y, x]} - \frac{M_G[z, y] \cdot M_G[x, t]}{M_G[x, y]}$$

$$M_{G \wedge xy}[x, t] := \frac{M_G[y, t]}{M_G[y, x]} \quad M_{G \wedge xy}[y, t] := \frac{\sigma(1) \cdot M_G[x, t]}{M_G[x, y]}$$

$$M_{G \wedge xy}[z, x] := \frac{\sigma(1) \cdot M_G[z, y]}{M_G[x, y]} \quad M_{G \wedge xy}[z, y] := \frac{M_G[z, x]}{M_G[y, x]}$$

$$M_{G \wedge xy}[x, y] := -\frac{1}{M_G[y, x]} \quad M_{G \wedge xy}[y, x] := -\frac{\sigma(1)^2}{M_G[x, y]}$$

A \mathbb{F}^* -graph H is pivot-equivalent to a \mathbb{F}^* -graph G if H is obtained by applying a sequence of pivot-complementations to G . We call H a pivot-minor of G if $H = G'[X]$ for some $X \subseteq V_G$ and G' pivot-equivalent to G . Moreover, H is a proper pivot-minor of G if $X \subsetneq V_G$.

Note that $G \wedge xy = G \wedge yx$ if $\sigma(1) = 1$. In the case of undirected graphs ($\mathbb{F} = \mathbb{F}_2$), this definition coincides with the pivot-complementation of undirected graphs [29]. The following lemma shows that this transformation is well defined.

Lemma 3.19 *Let G be a σ -symmetric \mathbb{F}^* -graph and let xy be an edge of G . Then $G \wedge xy$ is also σ -symmetric.*

Proof. Let $z, t \in V$, with $z \neq t$. If $\{z, t\} \cap \{x, y\} = \emptyset$, then

$$\begin{aligned}
M_{G \wedge xy}[t, z] &= M_G[t, z] - \frac{M_G[t, x] \cdot M_G[y, z]}{M_G[y, x]} - \frac{M_G[t, y] \cdot M_G[x, z]}{M_G[x, y]} \\
&= \sigma(M_G[z, t]) - \frac{\sigma(M_G[x, t]) \cdot \sigma(M_G[z, y])}{\sigma(M_G[x, y])} - \frac{\sigma(M_G[y, t]) \cdot \sigma(M_G[z, x])}{\sigma(M_G[y, x])} \\
&= \sigma(M_G[z, t]) - \sigma\left(\frac{M_G[x, t] \cdot M_G[z, y]}{M_G[x, y]}\right) - \sigma\left(\frac{M_G[y, t] \cdot M_G[z, x]}{M_G[y, x]}\right) \\
&= \sigma\left(M_G[z, t] - \frac{M_G[x, t] \cdot M_G[z, y]}{M_G[x, y]} - \frac{M_G[y, t] \cdot M_G[z, x]}{M_G[y, x]}\right) \\
&= \sigma(M_{G \wedge xy}[z, t]).
\end{aligned}$$

If $t \neq y$, then:

$$\begin{aligned}
M_{G \wedge xy}[t, x] &= \frac{\sigma(1) \cdot M_G[t, y]}{M_G[x, y]} = \frac{\sigma(1) \cdot \sigma(M_G[y, t])}{\sigma(M_G[y, x])} \\
&= \sigma\left(\frac{M_G[y, t]}{M_G[y, x]}\right) = \sigma(M_{G \wedge xy}[x, t]).
\end{aligned}$$

Finally:

$$\begin{aligned}
M_{G \wedge xy}[y, x] &= -\frac{\sigma(1)^2}{M_G[x, y]} = -\frac{\sigma(1)^2}{\sigma(M_G[y, x])} \\
&= \sigma\left(-\frac{1^2}{M_G[y, x]}\right) = \sigma(M_{G \wedge xy}[x, y]). \quad \square
\end{aligned}$$

Similarly to Lemma 3.17, the following lemma proves that pivot complementation does not increase \mathbb{F} -rank-width.

Lemma 3.20 *Let G be a σ -symmetric \mathbb{F}^* -graph and xy an edge of G . For every subset X of V_G :*

$$\mathbb{F}\text{-cutrk}_{G \wedge xy}(X) = \mathbb{F}\text{-cutrk}_G(X).$$

Proof. Let $Y := V_G \setminus X$. We can assume w.l.o.g. that $x \in X$. If $y \in X$, then (with $X' := X \setminus \{x, y\}$)

$$\begin{aligned}
\text{rk}(M_{G \wedge xy}[X, Y]) &= \text{rk} \begin{pmatrix} \frac{1}{M_G[y, x]} \cdot M_G[y, Y] \\ \frac{\sigma(1)}{M_G[x, y]} \cdot M_G[x, Y] \\ M_G[X', Y] - \frac{M_G[X', x] \cdot M_G[y, Y]}{M_G[y, x]} - \frac{M_G[X', y] \cdot M_G[x, Y]}{M_G[x, y]} \end{pmatrix} \\
&= \text{rk} \begin{pmatrix} \frac{1}{M_G[y, x]} \cdot M_G[y, Y] \\ \frac{\sigma(1)}{M_G[x, y]} \cdot M_G[x, Y] \\ M_G[X', Y] - \frac{M_G[X', x] \cdot M_G[y, Y]}{M_G[y, x]} \end{pmatrix} \\
&= \text{rk} \begin{pmatrix} \frac{1}{M_G[y, x]} \cdot M_G[y, Y] \\ \frac{\sigma(1)}{M_G[x, y]} \cdot M_G[x, Y] \\ M_G[X', Y] \end{pmatrix} = \text{rk} \begin{pmatrix} M_G[y, Y] \\ M_G[x, Y] \\ M_G[X', Y] \end{pmatrix} \\
&= \text{rk}(M_G[X, Y]).
\end{aligned}$$

If $y \notin X$, then (with $X' := X \setminus \{x\}$ and $Y' := Y \setminus \{y\}$)

$$\begin{aligned}
\text{rk}(M_{G \wedge xy}[X, Y]) &= \text{rk} \begin{pmatrix} -\frac{1}{M_G[y, x]} & \frac{M_G[y, Y']}{M_G[y, x]} \\ \frac{M_G[X', x]}{M_G[y, x]} & M_G[X', Y'] - \frac{M_G[X', x] \cdot M_G[y, Y']}{M_G[y, x]} - \frac{M_G[X', y] \cdot M_G[x, Y']}{M_G[x, y]} \end{pmatrix} \\
&= \text{rk} \begin{pmatrix} -\frac{1}{M_G[y, x]} & \frac{M_G[y, Y']}{M_G[y, x]} \\ 0 & M_G[X', Y'] - \frac{M_G[X', y] \cdot M_G[x, Y']}{M_G[x, y]} \end{pmatrix} \\
&= \text{rk} \begin{pmatrix} -\frac{1}{M_G[y, x]} & 0 \\ 0 & M_G[X', Y'] - \frac{M_G[X', y] \cdot M_G[x, Y']}{M_G[x, y]} \end{pmatrix} \\
&= \text{rk} \begin{pmatrix} M_G[x, y] & 0 \\ 0 & M_G[X', Y'] - \frac{M_G[X', y] \cdot M_G[x, Y']}{M_G[x, y]} \end{pmatrix} \\
&= \text{rk} \begin{pmatrix} M_G[x, y] & 0 \\ M_G[X', y] & M_G[X', Y'] - \frac{M_G[X', y] \cdot M_G[x, Y']}{M_G[x, y]} \end{pmatrix} \\
&= \text{rk} \begin{pmatrix} M_G[x, y] & M_G[x, Y'] \\ M_G[X', y] & M_G[X', Y'] \end{pmatrix} \\
&= \text{rk}(M_G[X, Y]). \quad \square
\end{aligned}$$

Proposition 3.21 *Let G and H be two σ -symmetric \mathbb{F}^* -graphs. If H is σ -locally-equivalent (resp. pivot-equivalent) to G , then the \mathbb{F} -rank-width of H is equal to the \mathbb{F} -rank-width of G . If H is a σ -vertex-minor (resp. pivot-minor)*

of G , then the \mathbb{F} -rank-width of H is at most the \mathbb{F} -rank-width of G .

Proof. The first statement is obvious by Lemma 3.17 and Lemma 3.20. Since taking sub-matrices does not increase the rank, it does not increase the \mathbb{F} -rank-width. So, the second statement is true. \square

Our goal now is to prove the following which is a generalization of Theorem 2.5.

Theorem 3.22 (i) For each positive integer $k \geq 1$, there is a set $\mathcal{C}_k^{(\mathbb{F}, \sigma)}$ of σ -symmetric \mathbb{F}^* -graphs, each having at most $(6^{k+1} - 1)/5$ vertices, such that a σ -symmetric \mathbb{F}^* -graph G has \mathbb{F} -rank-width at most k if and only if no σ -symmetric \mathbb{F}^* -graph in $\mathcal{C}_k^{(\mathbb{F}, \sigma)}$ is isomorphic to a pivot-minor of G .
(ii) Suppose that a σ -compatible $\lambda \in \mathbb{F}^*$ exists. Then for each positive integer $k \geq 1$, there is a set $\mathcal{C}'_k^{(\mathbb{F}, \sigma)}$ of σ -symmetric \mathbb{F}^* -graphs, each having at most $(6^{k+1} - 1)/5$ vertices, such that a σ -symmetric \mathbb{F}^* -graph G has \mathbb{F} -rank-width at most k if and only if no σ -symmetric \mathbb{F}^* -graph in $\mathcal{C}'_k^{(\mathbb{F}, \sigma)}$ is isomorphic to a σ -vertex-minor of G .

Note that $\mathcal{C}_k^{(\mathbb{F}, \sigma)}$ and $\mathcal{C}'_k^{(\mathbb{F}, \sigma)}$ are finite if \mathbb{F} is finite. For doing so we adapt the same techniques as in [21, 29]. We first prove some inequalities concerning cut-rank functions. The following one is a counterpart of [29, Proposition 4.3]. All the notions of linear algebra are borrowed from [26].

Proposition 3.23 Let G be a σ -symmetric \mathbb{F}^* -graph, λ a σ -compatible element in \mathbb{F}^* and x a vertex of G . For every subset X of $V_G \setminus \{x\}$,

$$\mathbb{F}\text{-cutrk}_{(G^*(x, \lambda)) - x}(X) = \text{rk} \begin{pmatrix} -1 & M_G[x, V_G \setminus (X \cup x)] \\ M_G[X, x] & M_G[X, V_G \setminus (X \cup x)] \end{pmatrix} - 1$$

Proof. Let X be a subset of $V_G \setminus \{x\}$ and let $Y := V_G \setminus (X \cup \{x\})$. We let J be the matrix $(M_G[z, x] \cdot M_G[x, t])_{z \in X, t \in Y}$. Then,

$$\begin{aligned} \mathbb{F}\text{-cutrk}_{(G^*(x, \lambda)) - x}(X) &= \text{rk}(M_{G^*(x, \lambda)}[X, Y]) \\ &= \text{rk}(M_G[X, Y] + \lambda \cdot J) \\ &= \text{rk} \underbrace{\begin{pmatrix} -1 \cdot \lambda^{-1} & M_G[x, Y] \\ 0 & M_G[X, Y] + \lambda \cdot J \end{pmatrix}}_A - 1 \end{aligned}$$

We now show how to transform the $(\{x\} \cup X, \{x\} \cup Y)$ -matrix A by using elementary row operations in order to get the desired equality. For each $z \in X$,

$$-\lambda \cdot M_G[z, x] \cdot A[x, Y \cup \{x\}] = \begin{pmatrix} M_G[z, x] & -\lambda \cdot J[z, Y] \end{pmatrix}.$$

Hence,

$$-\lambda \cdot M_G[z, x] \cdot A[x, Y \cup \{x\}] + A[z, Y \cup \{x\}] = \begin{pmatrix} M_G[z, x] & M_G[z, Y] \end{pmatrix}.$$

Therefore, by adding $-\lambda \cdot M_G[z, x] \cdot A[x, Y \cup \{x\}]$ to each row $A[z, Y \cup \{x\}]$ of A we get the matrix $\begin{pmatrix} -1 & M_G[x, Y] \\ M_G[X, x] & M_G[X, Y] \end{pmatrix}$. This concludes the proof. \square

The following lemma is thus the counterpart of [29, Lemma 4.4] and [21, Proposition 3.2].

Lemma 3.24 *Let G be a σ -symmetric \mathbb{F}^* -graph and x a vertex in V_G . Assume that (X_1, X_2) and (Y_1, Y_2) are partitions of $V_G \setminus \{x\}$. Then,*

$$\mathbb{F}\text{-cutrk}_{G-x}(X_1) + \mathbb{F}\text{-cutrk}_{(G*(x,\lambda))-x}(Y_1) \geq \mathbb{F}\text{-cutrk}_G(X_1 \cap Y_1) + \mathbb{F}\text{-cutrk}_G(X_2 \cap Y_2) - 1.$$

Proof. We recall that for every vertex z of G , $M_G[z, z] = 0$. Let M' be obtained from M_G by replacing $M_G[x, x]$ by -1 . It is worth noticing that for every subset X of V_G , $\text{rk}(M_G[X, V_G \setminus X]) = \text{rk}(M'[X, V_G \setminus X])$. We recall that $Y_2 = V_G \setminus (Y_1 \cup \{x\})$ and $X_2 = V_G \setminus (X_1 \cup \{x\})$. By definition of M' ,

$$M'[Y_1 \cup \{x\}, Y_2 \cup \{x\}] = \begin{pmatrix} -1 & M_G[x, Y_2] \\ M_G[Y_1, x] & M_G[Y_1, Y_2] \end{pmatrix}.$$

By Proposition 3.23,

$$\mathbb{F}\text{-cutrk}_{G-x}(X_1) + \mathbb{F}\text{-cutrk}_{(G*(x,\lambda))-x}(Y_1) = \text{rk}(M_G[X_1, X_2]) + \text{rk}(M'[Y_1 \cup \{x\}, Y_2 \cup \{x\}]) - 1.$$

Since $\text{rk}(M_G[X_1, X_2]) = \text{rk}(M'[X_1, X_2])$, by Proposition 3.9 we get the inequality

$$\begin{aligned} & \text{rk}(M_G[X_1, X_2]) + \text{rk}(M'[Y_1 \cup \{x\}, Y_2 \cup \{x\}]) \geq \\ & \text{rk}(M'[X_1 \cap Y_1, X_2 \cup Y_2 \cup \{x\}]) + \text{rk}(M'[X_1 \cup Y_1 \cup \{x\}, X_2 \cap Y_2]). \end{aligned}$$

Hence,

$$\mathbb{F}\text{-cutrk}_{G-x}(X_1) + \mathbb{F}\text{-cutrk}_{(G*(x,\lambda))-x}(Y_1) \geq \mathbb{F}\text{-cutrk}_G(X_1 \cap Y_1) + \mathbb{F}\text{-cutrk}_G(X_1 \cup Y_1 \cup x) - 1.$$

By the symmetry of $\mathbb{F}\text{-cutrk}_G$, we get the desired inequality. \square

Similarly, we get the followings for pivot-minor.

Proposition 3.25 *Let $G = (V, E)$ be a σ -symmetric \mathbb{F}^* -graph and xy an edge of G . For every subset X of $V_G \setminus \{x\}$,*

$$\mathbb{F}\text{-cutrk}_{(G \wedge xy)-x}(X) = \text{rk} \begin{pmatrix} 0 & M_G[x, V \setminus (X \cup x)] \\ M_G[X, x] & M_G[X, V \setminus (X \cup x)] \end{pmatrix} - 1$$

Proof. Suppose w.l.o.g. that $y \in X$ (otherwise replace X by $V_G \setminus (X \cup \{x\})$). Let $Y := V_G \setminus (X \cup \{x\})$ and $X' := X \setminus \{y\}$. Then, by elementary row and column operations, we have:

$$\begin{aligned}
\mathbb{F}\text{-cutrk}_{(G \wedge xy)-x}(X) &= \text{rk} \begin{pmatrix} \frac{\sigma(1)}{M_G[x,y]} \cdot M_G[x,Y] \\ M_G[X',Y] - \frac{M_G[X',x] \cdot M_G[y,Y]}{M_G[y,x]} - \frac{M_G[X',y] \cdot M_G[x,Y]}{M_G[x,y]} \end{pmatrix} \\
&= \text{rk} \begin{pmatrix} M_G[y,Y] \\ M_G[X',Y] - \frac{M_G[X',x] \cdot M_G[y,Y]}{M_G[y,x]} \end{pmatrix} \\
&= \text{rk} \begin{pmatrix} M_G[y,x] & M_G[y,Y] \\ 0 & M_G[y,Y] \\ 0 & M_G[X',Y] - \frac{M_G[X',x] \cdot M_G[y,Y]}{M_G[y,x]} \end{pmatrix} - 1 \\
&= \text{rk} \begin{pmatrix} 0 & M_G[y,Y] \\ M_G[y,x] & M_G[y,Y] \\ M_G[X',x] & M_G[X',Y] \end{pmatrix} - 1 \\
&= \text{rk} \begin{pmatrix} 0 & M_G[y,Y] \\ M_G[X,x] & M_G[X,Y] \end{pmatrix} - 1. \quad \square
\end{aligned}$$

Lemma 3.26 *Let G be a σ -symmetric \mathbb{F}^* -graph and xy an edge in V_G . Assume that (X_1, X_2) and (Y_1, Y_2) are partitions of $V_G \setminus \{x\}$. Then*

$$\mathbb{F}\text{-cutrk}_{G-x}(X_1) + \mathbb{F}\text{-cutrk}_{(G \wedge xy)-x}(Y_1) \geq \mathbb{F}\text{-cutrk}_G(X_1 \cap Y_1) + \mathbb{F}\text{-cutrk}_G(X_2 \cap Y_2) - 1.$$

Proof. We recall that $Y_2 = V_G \setminus (Y_1 \cup \{x\})$ and $X_2 = V_G \setminus (X_1 \cup \{x\})$. By definition of M ,

$$M[Y_1 \cup \{x\}, Y_2 \cup \{x\}] = \begin{pmatrix} 0 & M_G[x, Y_2] \\ M_G[Y_1, x] & M_G[Y_1, Y_2] \end{pmatrix}.$$

By Proposition 3.25,

$$\mathbb{F}\text{-cutrk}_{G-x}(X_1) + \mathbb{F}\text{-cutrk}_{(G \wedge xy)-x}(Y_1) = \text{rk}(M_G[X_1, X_2]) + \text{rk}(M[Y_1 \cup \{x\}, Y_2 \cup \{x\}]) - 1.$$

By Proposition 3.9 we get the inequality

$$\begin{aligned}
&\text{rk}(M_G[X_1, X_2]) + \text{rk}(M[Y_1 \cup \{x\}, Y_2 \cup \{x\}]) \geq \\
&\text{rk}(M[X_1 \cap Y_1, X_2 \cup Y_2 \cup \{x\}]) + \text{rk}(M[X_1 \cup Y_1 \cup \{x\}, X_2 \cap Y_2]).
\end{aligned}$$

Hence,

$$\mathbb{F}\text{-cutrk}_{G-x}(X_1) + \mathbb{F}\text{-cutrk}_{(G \wedge xy)-x}(Y_1) \geq \mathbb{F}\text{-cutrk}_G(X_1 \cap Y_1) + \mathbb{F}\text{-cutrk}_G(X_1 \cup Y_1 \cup x) - 1.$$

By the symmetry of \mathbb{F} -cutrk $_G$, we get the desired inequality. \square

The most important ingredients for proving Theorem 3.22 are Propositions 3.23 and 3.25, and Lemmas 3.24 and 3.26. All the other ingredients are already proved in [21,29] except that they are stated for the connectivity function of matroids in [21] and for undirected graphs in [29]. Their proofs rely only on the fact that the parameter is symmetric, submodular and integer valued. We include them for completeness. We first recall some definitions [21,29].

Let V be a finite set and $f : 2^V \rightarrow \mathbb{N}$ a symmetric and submodular function. Let (A, B) be a bipartition of V . A *branching* of B is a triple (T, r, \mathcal{L}) where T is a sub-cubic tree with a fixed node $r \in L_T$ and such that $(T-r, \mathcal{L})$ is a layout of B . For an edge e of T and a node v of T , we let T_{ev} be the set of nodes in the component of $T-e$ not containing v and we let $Y_{ev} := \mathcal{L}^{-1}(L_{T_{ev}})$. We say that B is *k-branched* if there exists a branching (T, r, \mathcal{L}) such that for each edge e of T , $f(Y_{ev}) \leq k$. It is worth noticing that if A and B are *k-branched*, then the f -width of V is at most k .

A subset A of V is called *titanic* with respect to f if for every partition (A_1, A_2, A_3) of A , there is a $i \in \{1, 2, 3\}$ such that $f(A_i) \geq f(A)$ (A_1, A_2 or A_3 may be empty).

The following lemma is proved in [29, Lemma 5.1] for cutrk $_G$, in [21, Lemma 2.1] for the connectivity function of matroids, and in [22, Lemma 3.3] for all symmetric and submodular functions.

Lemma 3.27 ([22, Lemma 3.3]) *Let V be a finite set and $f : 2^V \rightarrow \mathbb{N}$ a symmetric and submodular function. Assume that the f -width of V is at most k . Let (A, B) be a bipartition of V such that $f(A) \leq k$. If A is titanic with respect to f , then B is k -branched.*

Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be a function. A σ -symmetric \mathbb{F}^* -graph G is called (m, g) -connected if for every bipartition (A, B) of V_G , \mathbb{F} -cutrk $_G(A) = \ell < m$ implies $|A| \leq g(\ell)$ or $|B| \leq g(\ell)$. This notion will help to bound the order of the minimal σ -symmetric \mathbb{F}^* -graphs that every σ -symmetric \mathbb{F}^* -graph of \mathbb{F} -rank-width k must exclude as pivot-minor or σ -vertex-minors.

Lemma 3.28 *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a non-decreasing function with $f(0) = 0$. Let G be an (m, f) -connected σ -symmetric \mathbb{F}^* -graph and x a vertex of G . Then either $G-x$ or $(G \wedge xy)-x$ is $(m, 2f)$ -connected (for an edge xy). Moreover if a σ -compatible $\lambda \in \mathbb{F}^*$ exists, either $G-x$ or $(G * (x, \lambda))-x$ is $(m, 2f)$ -connected.*

Proof. Since $f(0) = 0$, G is connected. Let y be a neighbor of x . Suppose neither $G-x$ nor $(G \wedge xy)-x$ is $(m, 2f)$ -connected. Then there are bi-

partitions (A_1, A_2) and (B_1, B_2) of $V_G \setminus \{x\}$ such that $a = \mathbb{F}\text{-cutrk}_{G-x}(A_1)$, $b = \mathbb{F}\text{-cutrk}_{(G \wedge xy)-x}(B_1)$, and $|A_i| > 2f(a)$, $|B_i| > 2f(b)$ for $i = 1, 2$.

We may assume that $a \geq b$. By Lemma 3.26, we have

$$\mathbb{F}\text{-cutrk}_G(A_1 \cap B_1) + \mathbb{F}\text{-cutrk}_G(A_2 \cap B_2) \leq a + b + 1.$$

Thus, either $\mathbb{F}\text{-cutrk}_G(A_1 \cap B_1) \leq a$ or $\mathbb{F}\text{-cutrk}_G(A_2 \cap B_2) \leq b$. So, by hypothesis either $|A_1 \cap B_1| \leq f(a)$ or $|A_2 \cap B_2| \leq f(b)$. Assume that $|A_2 \cap B_2| \leq f(b)$. Similarly, we also have either $|A_2 \cap B_1| \leq f(a)$ or $|A_1 \cap B_2| \leq f(b)$. Since $|A_1 \cap B_2| = |B_2| - |B_2 \cap A_2| > f(b)$, we have $|A_2 \cap B_1| \leq f(a)$. Then $|A_2| = |A_2 \cap B_1| + |A_2 \cap B_2| \leq f(a) + f(b) \leq 2f(a)$. This yields a contradiction.

The proof of the second statement is similar, using Lemma 3.24. \square

We let $g(n) = (6^n - 1)/5$. Note that $g(0) = 0$, $g(1) = 1$ and $g(n) = 6g(n-1) + 1$ for all $n \geq 1$. We now prove that the minimal σ -symmetric \mathbb{F}^* -graphs that have \mathbb{F} -rank-width at least $k + 1$ are $(k + 1, g)$ -connected.

Lemma 3.29 *Let $k \geq 1$ and let G be a σ -symmetric \mathbb{F}^* -graph of \mathbb{F} -rank-width larger than k . If every proper pivot-minor of G has \mathbb{F} -rank-width at most k , then G is $(k + 1, g)$ -connected. Similarly, if a σ -compatible $\lambda \in \mathbb{F}^*$ exists, and every proper σ -vertex-minor of G has \mathbb{F} -rank-width at most k , then G is $(k + 1, g)$ -connected.*

Proof. The proof is similar to the one of [29, Lemma 5.3]. We assume that G is connected since the \mathbb{F} -rank-width of G is the maximum of the \mathbb{F} -rank-width of its connected components. It is now easy to see that G is $(1, g)$ -connected.

Assume that $m \leq k$ and that G is (m, g) -connected but G is not $(m + 1, g)$ -connected. Then there exists a bipartition (A, B) with $\mathbb{F}\text{-cutrk}_G(A) = m$ such that $|A| > g(m)$ and $|B| > g(m)$. Also, either A or B is not k -branched ($\mathbb{F}\text{-rwd}(G) > k$). We may assume that B is not k -branched. Let $x \in A$ and $xy \in E_G$.

By Lemma 3.28, either $G-x$ or $(G \wedge xy)-x$ is $(m, 2g)$ -connected; assume $G-x$ is $(m, 2g)$ -connected. Since $G-x$ and $(G \wedge xy)-x$ are proper pivot-minors of G , they both have \mathbb{F} -rank-width at most k . Let (A_1, A_2, A_3) be a tri-partition of $A \setminus \{x\}$. Since $|A| > g(m) = 6g(m-1) + 1$, there exists an $i \in [3]$ such that $|A_i| > 2g(m-1)$. Since $G-x$ is $(m, 2g)$ -connected and $|A_i| > 2g(m-1)$,

$$\mathbb{F}\text{-cutrk}_{G-x}(A_i) \geq m \geq \mathbb{F}\text{-cutrk}_{G-x}(A \setminus \{x\}).$$

Therefore, by Lemma 3.27 B is k -branched in $G-x$. Since B is not k -branched in G , there exists $W \subseteq B$ such that

$$\mathbb{F}\text{-cutrk}_G(W) = \mathbb{F}\text{-cutrk}_{G-x}(W) + 1.$$

Thus, the column vectors of $M_G[W, V_G \setminus (W \cup \{x\})]$ do not span $M_G[W, x]$. So, the column vectors of $M_G[W, V_G \setminus (B \cup \{x\})]$ do not span $M_G[W, x]$. Hence, the column vectors of $M_G[B, V_G \setminus (B \cup \{x\})]$ do not span $M_G[B, x]$. Therefore,

$$\mathbb{F}\text{-cutrk}_{G-x}(B) = \mathbb{F}\text{-cutrk}_G(B) - 1 = m - 1.$$

This implies that $|B| \leq 2g(m - 1)$ or $|A \setminus \{x\}| \leq 2g(m - 1)$. A contradiction.

The proof of the second statement is similar (replace $G \wedge xy$ by $G * (x, \lambda)$). \square

As a consequence of Lemma 3.29, we get the following.

Theorem 3.30 (Size of Excluded Pivot-Minor and σ -Vertex-Minors)

Let $k \geq 1$ and let G be a σ -symmetric \mathbb{F}^ -graph. If G has \mathbb{F} -rank-width larger than k but every proper pivot-minor of G has \mathbb{F} -rank-width at most k , then $|V_G| \leq (6^{k+1} - 1)/5$.*

Moreover, if a σ -compatible $\lambda \in \mathbb{F}^$ exists, and if G has \mathbb{F} -rank-width larger than k but every proper σ -vertex-minor of G has \mathbb{F} -rank-width at most k , then $|V_G| \leq (6^{k+1} - 1)/5$.*

Proof. Let $x \in V_G$. We may assume that $G-x$ is $(k + 1, 2g)$ -connected by Lemmas 3.28 and 3.29. Since $G-x$ has \mathbb{F} -rank-width k , there exists a bipartition (A, B) of $V_G \setminus \{x\}$ such that $|A| \geq \frac{1}{3}(|V_G| - 1)$ and $|B| \geq \frac{1}{3}(|V_G| - 1)$ and $\mathbb{F}\text{-cutrk}_{G-x}(A) \leq k$. By $(k + 1, 2g)$ -connectivity, either $|A| \leq 2g(k)$ or $|B| \leq 2g(k)$. Therefore, $|V_G| - 1 \leq 6g(k)$ and consequently $|V_G| \leq 6g(k) + 1 = g(k + 1)$. \square

It is surprising that the bound $(6^{k+1} - 1)/5$ does not depend neither on \mathbb{F} nor on σ . But that is because the proof technique is based on the \mathbb{F} -cutrk $_G$ -width of V_G and neither on \mathbb{F} nor on σ . However, the \mathbb{F} -rank-width depends on \mathbb{F} since there is no reason that the rank of a matrix is the same in two different fields. But, as we will see in the following proof of Theorem 3.22, the set of σ -symmetric \mathbb{F}^* -graphs to exclude as pivot-minors and σ -vertex-minor depends on \mathbb{F} and σ .

Proof of Theorem 3.22. We show only the proof for the first statement. the other proof is similar. If $k < 0$, we let $\mathcal{C}_k^{(\mathbb{F}, \sigma)} = \emptyset$. If $k = 0$, we let $\mathcal{C}_0^{(\mathbb{F}, \sigma)} :=$

$\{\mathbf{a} \mid a \in \mathbb{F}^*\}$ where \mathbf{a} is the σ -symmetric \mathbb{F}^* -graph $(\{x, y\}, \{x \xrightarrow{a} y, y \xrightarrow{\sigma(a)} x\})$. It is clear that G has \mathbb{F} -rank-width at most 0 if and only if G has no pivot-minor isomorphic to any $\mathbf{a} \in \mathcal{C}_0^{(\mathbb{F}, \sigma)}$.

Assume now that $k \geq 1$ and let $\mathcal{C}_k^{(\mathbb{F}, \sigma)}$ be the set, up to isomorphism, of σ -symmetric \mathbb{F}^* -graphs H such that $\mathbb{F}\text{-rwd}(H) > k$ and every proper pivot-minor of H has \mathbb{F} -rank-width at most k . By Theorem 3.30, each σ -symmetric \mathbb{F}^* -graph in $\mathcal{C}_k^{(\mathbb{F}, \sigma)}$ has at most $(6^{k+1} - 1)/5$ vertices.

Let G be a σ -symmetric \mathbb{F}^* -graph of \mathbb{F} -rank-width at most k . Since every \mathbb{F}^* -graph in $\mathcal{C}_k^{(\mathbb{F}, \sigma)}$ has \mathbb{F} -rank-width larger than k , no \mathbb{F}^* -graph in $\mathcal{C}_k^{(\mathbb{F}, \sigma)}$ is isomorphic to a pivot-minor of G .

Conversely, assume that the \mathbb{F} -rank-width of G is larger than k and let H be a proper pivot-minor of G of minimum size such that $\mathbb{F}\text{-rwd}(H) > k$. Then there exists a \mathbb{F}^* -graph $H' \in \mathcal{C}_k^{(\mathbb{F}, \sigma)}$ isomorphic to H . \square

Moreover, using the characterization of \mathbb{F}^* -graphs of \mathbb{F} -rank-width 1 [23,24], obstructions for \mathbb{F}^* -graphs of \mathbb{F} -rank-width 1 by vertex-minor (resp. pivot-minor) have at most 5 (resp. 6) vertices. In [30], Oum derives from the *principal pivot transformation* of Tucker (see [36] for instance) a notion of *pivot-minor* for symmetric and skew-symmetric matrices and proved that symmetric and skew-symmetric matrices of bounded rank-width are well-quasi-ordered by this relation. Our notion of pivot-minor is a special case of Oum's notion when $\sigma(x) := x$ or $\sigma(x) := -x$. Hence, oriented graphs of bounded rank-width are well-quasi-ordered by pivot-minor. We generalise Oum's result to σ -symmetric matrices in [25].

3.3 Recognizing \mathbb{F} -Rank-Width at Most k

We give in this section a cubic-time algorithm that decides whether a \mathbb{F}^* -graph has \mathbb{F} -rank-width at most k , for fixed finite field \mathbb{F} and a fixed k . This algorithm is an easy corollary of the one by Hliněný and Oum concerning representable matroids [22]. We recall the necessary materials about matroids. We refer to Schrijver [35] for our matroid terminology. We let \mathbb{F} be a fixed finite field and $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ a sesqui-morphism.

Definition 3.31 (Matroids) *A pair $\mathcal{M} = (S, \mathcal{I})$ is called a matroid if S is a finite set and \mathcal{I} is a nonempty collection of subsets of S satisfying the following conditions*

(M1) *if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$,*

(M2) if $I, J \in \mathcal{I}$ and $|I| < |J|$, then $I \cup \{z\} \in \mathcal{I}$ for some $z \in J \setminus I$.

For $U \subseteq S$, a subset B of U is called a *base* of U if B is an inclusionwise maximal subset of U and belongs to \mathcal{I} . It is easy to see that, if B_1 and B_2 are bases of $U \subseteq S$, then B_1 and B_2 have the same size. The common size of the bases of a subset U of S is called the *rank* of U , denoted by $r_{\mathcal{M}}(U)$. A set $B \subseteq S$ is a *base* of \mathcal{M} if it is a base of S .

Let A be a $m \times n$ -matrix. Let $S := \{1, \dots, n\}$ and let \mathcal{I} be the collection of all those subsets I of S such that the columns of A with index in \mathcal{I} are linearly independent. Then $\mathcal{M} := (S, \mathcal{I})$ is a matroid. If A has entries in \mathbb{F} , then \mathcal{M} is said *representable* over \mathbb{F} and A is called a *representation* of \mathcal{M} over \mathbb{F} .

We now define the *branch-width* of matroids. Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. We let $\lambda_{\mathcal{M}}$ be defined such that for every subset U of S , $\lambda_{\mathcal{M}}(U) = r_{\mathcal{M}}(U) + r_{\mathcal{M}}(S \setminus U) - r_{\mathcal{M}}(S) + 1$ and call it the *connectivity function* of \mathcal{M} . The function $\lambda_{\mathcal{M}}$ is symmetric and submodular [35]. The *branch-width* of \mathcal{M} , denoted by $\text{bwd}(\mathcal{M})$, is the $\lambda_{\mathcal{M}}$ -width of S .

Definition 3.32 (Partitioned Matroids [22]) Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and \mathcal{P} a partition of S . The couple $(\mathcal{M}, \mathcal{P})$ is called a *partitioned matroid*. A partitioned matroid $(\mathcal{M}, \mathcal{P})$ is *representable* over \mathbb{F} if \mathcal{M} is representable over \mathbb{F} . For a partitioned matroid $(\mathcal{M}, \mathcal{P})$, we let $\lambda_{\mathcal{M}}^{\mathcal{P}}$ be defined such that for every $Z \subseteq \mathcal{P}$, we have $\lambda_{\mathcal{M}}^{\mathcal{P}}(Z) := \lambda_{\mathcal{M}}(\bigcup_{Y \in Z} Y)$. The *branch-width* of $(\mathcal{M}, \mathcal{P})$, denoted by $\text{bwd}(\mathcal{M}, \mathcal{P})$, is the $\lambda_{\mathcal{M}}^{\mathcal{P}}$ -width of \mathcal{P} .

We recall the following important result by Hliněný and Oum [22].

Theorem 3.33 ([22]) Let \mathbb{F} be a fixed finite field, and k be a fixed positive integer. There exists a cubic-time algorithm that takes as input a representable partitioned matroid $(\mathcal{M}, \mathcal{P})$ over \mathbb{F} given with the representation of \mathcal{M} over \mathbb{F} and outputs a layout of \mathcal{P} of $\lambda_{\mathcal{M}}^{\mathcal{P}}$ -width at most k or confirms that the branch-width of $(\mathcal{M}, \mathcal{P})$ is strictly greater than k .

We can now derive our recognition algorithm from Theorem 3.33. For that we borrow ideas from [22]. For a set X , we let X' be a disjoint copy of it defined as $\{x' \mid x \in X\}$. For G a \mathbb{F}^* -graph, we let \mathcal{M}_G be the matroid on $V_G \cup V'_G$ represented by the $(V_G, V_G \cup V'_G)$ -matrix (recall that I_n denotes the identity square matrix of size n):

$$V_G \begin{pmatrix} V_G & V'_G \\ I_{|V_G|} & M_G \end{pmatrix}$$

For each $x \in V$, we let $P_x := \{x, x'\}$ and we let $\Pi(G) := \{P_x \mid x \in V_G\}$. We now prove the following which is a counterpart of [29, Proposition 3.1].

Proposition 3.34 *Let G be a \mathbb{F}^* -graph. For every $X \subseteq V_G$, $\lambda_{\mathcal{M}_G}^{\Pi(G)}(P) = \text{rk}(M_G[X, V_G \setminus X]) + \text{rk}(M_G[V_G \setminus X, X]) + 1$ where $P := \{P_x \mid x \in X\}$.*

Proof. For $X \subseteq V_G$ and $P := \{P_x \mid x \in X\}$, we have

$$\begin{aligned} \lambda_{\mathcal{M}_G}^{\Pi(G)}(P) &= r_{\mathcal{M}_G}(X \cup X') + r_{\mathcal{M}_G}(V_G \setminus X \cup (V_G \setminus X)') - r_{\mathcal{M}_G}(V_G \cup V_G') + 1 \\ &= \text{rk} \begin{pmatrix} 0 & M_G[V_G \setminus X, X] \\ I_{|X|} & M_G[X, X] \end{pmatrix} + \text{rk} \begin{pmatrix} 0 & M_G[X, V_G \setminus X] \\ I_{|V_G| - |X|} & M_G[V_G \setminus X, V_G \setminus X] \end{pmatrix} - |V_G| + 1 \\ &= |X| + \text{rk}(M_G[V_G \setminus X, X]) + |V_G - X| + \text{rk}(M_G[X, V_G \setminus X]) - |V_G| + 1 \\ &= \text{rk}(M_G[X, V_G \setminus X]) + \text{rk}(M_G[V_G \setminus X, X]) + 1. \quad \square \end{aligned}$$

Since when G is σ -symmetric, we have $\text{rk}(M_G[X, V_G \setminus X]) = \text{rk}(M_G[V_G \setminus X, X]) = \mathbb{F}\text{-cutrk}_G(X)$, we get the followings as corollaries of Proposition 3.34.

Corollary 3.35 *Let G be a σ -symmetric \mathbb{F}^* -graph. For every $X \subseteq V_G$, $\lambda_{\mathcal{M}_G}^{\Pi(G)}(P) = 2 \cdot \mathbb{F}\text{-cutrk}_G(X) + 1$ where $P := \{P_x \mid x \in X\}$.*

Corollary 3.36 *Let G be a σ -symmetric \mathbb{F}^* -graph and let $p : V_G \rightarrow \Pi(G)$ be the bijective function such that $p(x) = P_x$. If (T, \mathcal{L}) is a layout of $\Pi(G)$ of $\lambda_{\mathcal{M}_G}^{\Pi(G)}$ -width $2k + 1$, then $(T, \mathcal{L} \circ p)$ is a layout of V_G of $\mathbb{F}\text{-cutrk}_G$ -width k . Conversely, if (T, \mathcal{L}) is a layout of V_G of $\mathbb{F}\text{-cutrk}_G$ -width k , then $(T, \mathcal{L} \circ p^{-1})$ is a layout of $\Pi(G)$ of $\lambda_{\mathcal{M}_G}^{\Pi(G)}$ -width $2k + 1$.*

Theorem 3.37 (Checking \mathbb{F} -Rank-Width at most k) *For fixed k and a fixed finite field \mathbb{F} , there exists a cubic-time algorithm that, for a σ -symmetric \mathbb{F}^* -graph G , either outputs a layout of V_G of $\mathbb{F}\text{-cutrk}_G$ -width at most k or confirms that the \mathbb{F} -rank-width of G is larger than k .*

Proof. Let k be fixed and let \mathcal{A} be the algorithm constructed in Theorem 3.33 for $2k + 1$. Let G be a σ -symmetric \mathbb{F}^* -graph. We run the algorithm \mathcal{A} with input $(\mathcal{M}_G, \Pi(G))$. If it confirms that $\text{bwd}(\mathcal{M}_G, \Pi(G)) > 2k + 1$, then the \mathbb{F} -rank-width of G is greater than k (Corollary 3.35). If it outputs a layout of $\Pi(G)$ of $\lambda_{\mathcal{M}_G}^{\Pi(G)}$ -width at most $2k + 1$, we can transform it into a layout of V_G of $\mathbb{F}\text{-cutrk}_G$ -width at most k by Corollary 3.36. The fact that the algorithm \mathcal{A} runs in cubic-time concludes the proof. \square

3.4 Specialisations to Graphs

We specialise in this section the \mathbb{F} -rank-width to directed and oriented graphs. As we already said, for undirected graphs the \mathbb{F}_2 -rank-width matches with the rank-width.

Directed Graphs. We recall that the adjacency matrix of a directed graph G is the (V_G, V_G) -matrix M_G over \mathbb{F}_2 where $M_G[x, y] := 1$ if and only if $(x, y) \in E_G$. This matrix is not symmetric except when G is undirected. In particular, $\text{rk}(M_G[X, V_G \setminus X])$ is *a priori* different from $\text{rk}(M_G[V_G \setminus X, X])$. The quest for finding another representation of directed graphs by matrices where $\text{rk}(M_G[X, V_G \setminus X]) = \text{rk}(M_G[V_G \setminus X, X])$ motivates the definition of sigma-symmetry. We now give this representation.

We recall that \mathbb{F}_4 is the finite field of order four. We let $\{0, 1, \varnothing, \varnothing^2\}$ be its elements with the property that $1 + \varnothing + \varnothing^2 = 0$ and $\varnothing^3 = 1$. Moreover, it is of characteristic 2. We let $\sigma_4 : \mathbb{F}_4 \rightarrow \mathbb{F}_4$ be the automorphism where $\sigma_4(\varnothing) = \varnothing^2$ and $\sigma_4(\varnothing^2) = \varnothing$. It is clearly a sesqui-morphism.

For every directed graph G , let $\tilde{G} := (V_G, E_G \cup \{(y, x) | (x, y) \in E_G\}, \ell_G)$ be the \mathbb{F}_4^* -graph where for every pair of vertices (x, y) :

$$\ell_G((x, y)) := \begin{cases} 1 & \text{if } (x, y) \in E_G \text{ and } (y, x) \in E_G, \\ \varnothing & (x, y) \in E_G \text{ and } (y, x) \notin E_G, \\ \varnothing^2 & (y, x) \in E_G \text{ and } (x, y) \notin E_G, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that \tilde{G} is σ_4 -symmetric and is actually the one constructed in Section 3. We define the rank-width of a directed graph G , denoted by $\mathbb{F}_4\text{-rwd}(G)$, as the \mathbb{F}_4 -rank-width of \tilde{G} .

Remark 3.38 Let G be an undirected graph. We denote by \vec{G} the directed graph obtained from G by replacing each edge xy in G by two opposite. By the definition of \vec{G} we have $A_G = M_{\vec{G}}$. Then $\mathbb{F}_4\text{-rwd}(\vec{G}) = \text{rwd}(G)$ since \mathbb{F}_4 is an extension of \mathbb{F}_2 .

We now specialise the notion of vertex-minor. We recall that given a sesqui-morphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$, an element λ of \mathbb{F}^* is said σ -compatible if $\sigma(\lambda) = \lambda \cdot \sigma(1)^2$. Since $\sigma_4(1) = 1$, 1 is σ_4 -compatible and is the only one. We then denote $G * v = G * (v, 1)$, and say that a directed graph H is a *vertex-minor* of a directed graph G if \tilde{H} is a vertex-minor of \tilde{G} . One easily verifies that if a

directed graph H is obtained from a directed graph G by applying a 1-local-complementation at x , then H is obtained from G by modifying the subgraph induced on the neighbours of x as shown on Table 1. Figure 1 gives an example of a 1-local complementation. In Figure 2 (resp. Figure 3), we give a set of obstructions for directed graphs of \mathbb{F}_4 -rank-width 1 with respect to vertex-minor relation (resp. pivot-minor relation).

G	$G * x$
$z \perp t$	$z \leftrightarrow t$
$z \rightarrow t$	$z \leftarrow t$
$z \leftarrow t$	$z \rightarrow t$
$z \leftrightarrow t$	$z \perp t$

(a)
(b)

Table 1

We use the following notations: $x \rightarrow y$ means $\ell_G((x, y)) = \varnothing$, $x \leftarrow y$ means $\ell_G((x, y)) = \varnothing^2$, $x \leftrightarrow y$ means $\ell_G((x, y)) = 1$, and $z \perp t$ means $\ell_G((x, y)) = 0$.

(a) Uniform Case: $z \leftarrow x \rightarrow t$ or $z \rightarrow x \leftarrow t$ or $z \leftrightarrow x \leftrightarrow t$.

(b) Non Uniform Case: $z \leftarrow x \leftarrow t$ or $z \rightarrow x \leftrightarrow t$ or $z \leftrightarrow x \rightarrow t$.

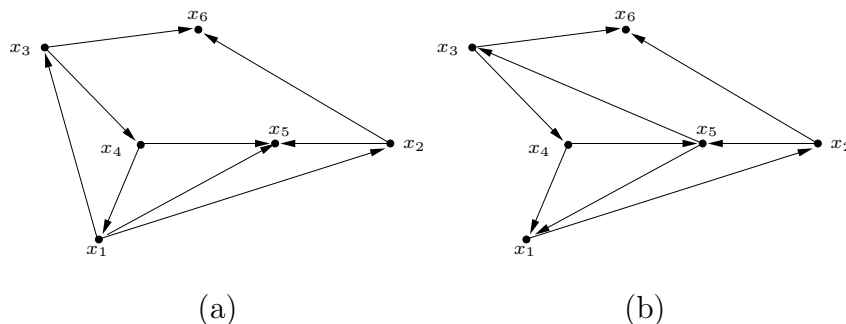


Figure 1. (a) A directed graph G . (b) The directed graph $G * x_4$.

Moreover, as in the undirected case, we have $G \wedge xy = G \wedge yx = G * x * y * x = G * y * x * y$. As corollaries of Theorem 3.22 and 3.37 we get the followings.

Theorem 3.39 *For each positive integer k , there is a finite list \mathcal{C}_k of directed graphs having at most $(6^{k+1} - 1)/5$ vertices such that a directed graph G has rank-width at most k if and only if no directed graph in \mathcal{C}_k is isomorphic to a vertex-minor of G .*

Theorem 3.40 *For fixed k , there exists a cubic-time algorithm that, for a directed graph G , either outputs a layout of V_G of \mathbb{F}_4 -cutrk $_G$ -width at most k or confirms that the rank-width of G is larger than k .*

Oriented Graphs. We can define another parameter in the case of oriented graphs. Let $G = (V, A)$ be an oriented graph, and let $\tilde{G} = (V, E, \ell)$ be the

Figure 2. Vertex-minor exclusions for directed graphs of \mathbb{F}_4 -rank-width 1.

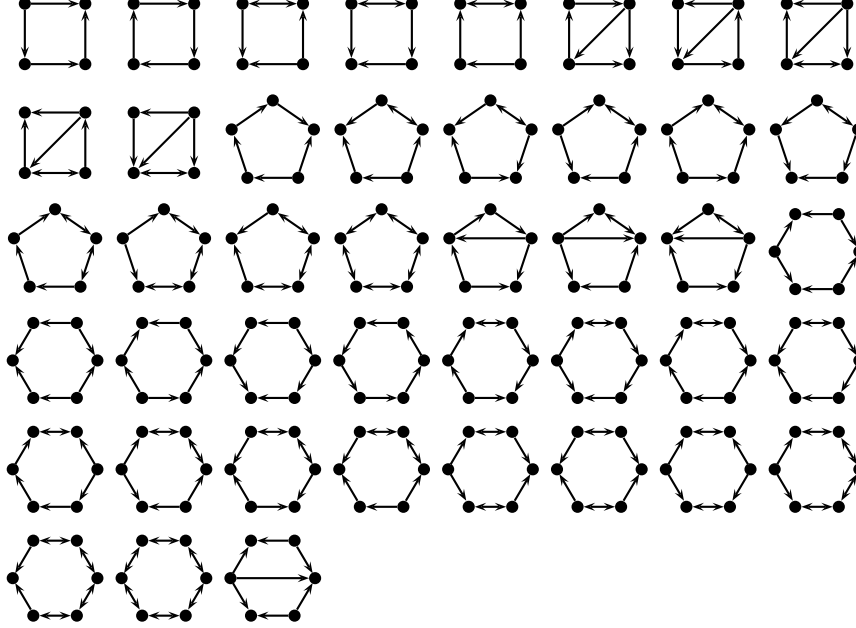


Figure 3. Pivot-minor exclusions for directed graphs of \mathbb{F}_4 -rank-width 1.

\mathbb{F}_3^* -graph such that $E = A \cup A'$ where $A' = \{(y, x) | (x, y) \in A\}$, $\ell((x, y)) := 1$ if $(x, y) \in A$ and $\ell((x, y)) := -1$ if $(x, y) \in A'$. Clearly, \tilde{G} is a σ -symmetric \mathbb{F}_3^* -graph, with $\sigma(x) := -x$. Moreover, one can show immediately that σ is a sesqui-morphism. Note that there is no σ -compatible λ in \mathbb{F}_3^* , thus no σ -local-complementation is defined on σ -symmetric \mathbb{F}_3^* -graphs. Nevertheless, oriented graphs of \mathbb{F}_3 -rank-width k are characterized by a finite set of oriented graphs $\mathcal{C}_k^{(\mathbb{F}_3, \sigma)}$ of forbidden pivot-minors (whereas sets $\mathcal{C}_k^{(\mathbb{F}_4, \sigma)}$ and $\mathcal{C}_k^{(\mathbb{F}_4, \sigma)}$ contains directed graphs). In Figure 5, we give a set of obstructions for oriented graphs of \mathbb{F}_3 -rank-width 1 with respect to pivot-minor relation.

\mathbb{F}_3 -rank-width and \mathbb{F}_4 -rank-width of oriented graphs are two equivalent parameters, since they are both equivalent to the clique width. But these two rank parameters are not equal. In one hand, tournaments of \mathbb{F}_3 -rank-width 1 are exactly tournaments completely decomposable by *bi-join decomposition* (see [24]), and a cut $\{X, Y\}$ in a tournament has \mathbb{F}_4 -rank 1 if and only if X or Y is a module. Since there are tournaments completely decomposable by bi-join and prime w.r.t. the modular decomposition (see [1]), there are oriented graphs of \mathbb{F}_3 -rank-width 1 and \mathbb{F}_4 -rank-width at least 2. On the other hand, the graph on Figure 4 (right) has \mathbb{F}_4 -rank-width 2 and \mathbb{F}_3 -rank-width 3.

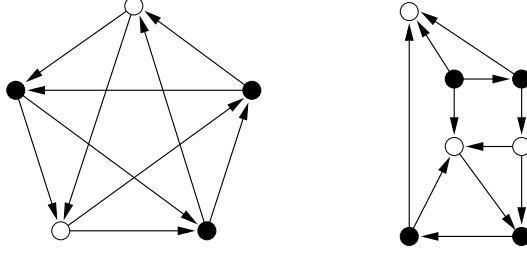


Figure 4. Left: an oriented graph of \mathbb{F}_3 -rank-width 1 and \mathbb{F}_4 -rank-width 2 (white/black vertices give a cut of \mathbb{F}_3 -rank-width 1). Right: an oriented graph of \mathbb{F}_3 -rank-width 3 and \mathbb{F}_4 -rank-width 2 (white/black vertices give a cut of \mathbb{F}_4 -rank-width 2).

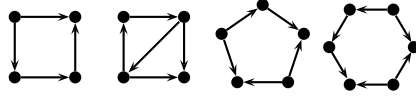


Figure 5. Pivot-minor exclusions for oriented graphs of \mathbb{F}_3 -rank-width 1.

4 The Second Notion of Rank-Width: \mathbb{F} -Bi-Rank-Width

In Section 4.1 we define the notion of \mathbb{F} -bi-rank-width for \mathbb{F}^* -graphs, sigma-symmetric or not, and compare it to clique-width and \mathbb{F} -rank-width. A cubic-time algorithm for recognising \mathbb{F}^* -graphs of \mathbb{F} -bi-rank-width at most k is presented in Section 4.2. A specialisation to graphs without colors on edges is given in Section 4.3.

4.1 Definitions and Comparisons to Other Parameters

Recall that if G is a \mathbb{F}^* -graph, we denote by M_G the (V_G, V_G) -matrix over \mathbb{F} where

$$M_G[x, y] := \begin{cases} \ell_G((x, y)) & \text{if } x \neq y, \\ 0 & \text{otherwise.} \end{cases}$$

As for the notion of \mathbb{F} -rank-width, we use matrix rank functions for the notion of \mathbb{F} -bi-rank-width.

Definition 4.1 (Bi-Cut-Rank Function) *For a \mathbb{F}^* -graph G , we let \mathbb{F} -bicutr $_G : 2^{V_G} \rightarrow \mathbb{N}$ where \mathbb{F} -bicutr $_G(X) = \text{rk}(M_G[X, V_G \setminus X]) + \text{rk}(M_G[V_G \setminus X, X])$ for all $X \subseteq V_G$.*

Lemma 4.2 *For every \mathbb{F}^* -graph G , the function \mathbb{F} -bicutr $_G$ is symmetric and*

submodular.

Proof. Let X and Y be subsets of V_G . We let $A_1 := M_G[X, V_G \setminus X]$, $A_2 := M_G[V_G \setminus X, X]$, $B_1 := M_G[Y, V_G \setminus Y]$ and $B_2 := M_G[V_G \setminus Y, Y]$. By definition,

$$\mathbb{F}\text{-bicutr}_G(X) = \text{rk}(A_1) + \text{rk}(A_2) = \text{rk}(A_2) + \text{rk}(A_1) = \mathbb{F}\text{-bicutr}_G(V_G \setminus X).$$

For the submodularity, we have by definition,

$$\mathbb{F}\text{-bicutr}_G(X) + \mathbb{F}\text{-bicutr}_G(Y) = \text{rk}(A_1) + \text{rk}(A_2) + \text{rk}(B_1) + \text{rk}(B_2)$$

By Proposition 3.9,

$$\text{rk}(A_1) + \text{rk}(B_1) \geq \text{rk}(M_G[X \cup Y, V_G \setminus X \cap V_G \setminus Y]) + \text{rk}(M_G[X \cap Y, V_G \setminus X \cup V_G \setminus Y])$$

and

$$\text{rk}(A_2) + \text{rk}(B_2) \geq \text{rk}(M_G[V_G \setminus X \cup V_G \setminus Y, X \cap Y]) + \text{rk}(M_G[V_G \setminus X \cap V_G \setminus Y, X \cup Y]).$$

Since $V_G \setminus X \cap V_G \setminus Y = V_G \setminus (X \cup Y)$ and $V_G \setminus X \cup V_G \setminus Y = V_G \setminus (X \cap Y)$ the second statement holds. \square

Definition 4.3 (\mathbb{F} -bi-rank-width) *The \mathbb{F} -bi-rank-width of a \mathbb{F}^* -graph, denoted by $\mathbb{F}\text{-brwd}(G)$, is the $\mathbb{F}\text{-bicutr}_G$ -width of V_G .*

The following proposition compares clique-width and \mathbb{F} -bi-rank-width when \mathbb{F} is finite and of order q . Its proof is easy.

Proposition 4.4 *For every \mathbb{F}^* -graph G , $\frac{1}{2}\mathbb{F}\text{-brwd}(G) \leq \text{cwd}(G) \leq 2 \cdot q^{\mathbb{F}\text{-brwd}(G)} - 1$.*

The following compares \mathbb{F} -bi-rank-width and \mathbb{F} -rank-width. Let $[G \mapsto \tilde{G}]$ be a mapping from $\mathcal{S}(\mathbb{F})$ to $\mathcal{S}(\mathbb{F}^2, \tilde{\sigma})$ such that for every $x, y \in V_G$, $M_{\tilde{G}}[x, y] = \gamma \cdot M_G[x, y] + \tau \cdot M_G[y, x]$ for fixed $\gamma, \tau \in (\mathbb{F}^2)^*$ with $\gamma/\tau \notin \mathbb{F}$. We recall that the mapping constructed in Section 3 respects this property.

Proposition 4.5 *Let G be a \mathbb{F}^* -graph. Then*

- (1) $\mathbb{F}^2\text{-rwd}(\tilde{G}) \leq \mathbb{F}\text{-brwd}(G) \leq 4 \cdot \mathbb{F}^2\text{-rwd}(\tilde{G})$.
- (2) *If G is σ -symmetric for some sesqui-morphism $\sigma : \mathbb{F} \rightarrow \mathbb{F}$, then $\mathbb{F}\text{-brwd}(G) = 2 \cdot \mathbb{F}\text{-rwd}(G)$.*

Before proving the proposition, we recall some technical properties about ranks of matrices. See the following books for more informations [26, 27].

Lemma 4.6 (i) *Let M be a matrix over \mathbb{F} . If the rank of M over \mathbb{F} is k , then the rank of M over any finite extension of \mathbb{F} is k .*

(ii) If A and B are two matrices over \mathbb{F} , then $\text{rk}(A + B) \leq \text{rk}(A) + \text{rk}(B)$ and $\text{rk}(A \cdot B) \leq \min\{\text{rk}(A), \text{rk}(B)\}$. If $a \in \mathbb{F}^*$, then $\text{rk}(a \cdot A) = \text{rk}(A)$.

By definition of \tilde{G} , we have:

Proposition 4.7 For every \mathbb{F}^* -graph G and every subset X of V_G , we have

$$M_{\tilde{G}}[X, V_G \setminus X] = \gamma \cdot M_G[X, V_G \setminus X] + \tau \cdot M_G^T[V_G \setminus X, X].$$

Proof of Proposition 4.5. It is sufficient to compare \mathbb{F}^2 -cutrk $_{\tilde{G}}(X)$ and \mathbb{F} -bicutr $k_G(X)$ for every subset X of V_G .

(1) From Lemma 4.6 and Proposition 4.7 we have:

$$\begin{aligned} \text{rk}(M_{\tilde{G}}[X, V_G \setminus X]) &\leq \text{rk}(M_G[X, V_G \setminus X]) + \text{rk}(M_G[V_G \setminus X, X]) \\ &= \mathbb{F}\text{-bicutr}k_G(X). \end{aligned}$$

We now prove that $\mathbb{F}\text{-bicutr}k_G(X) \leq 4 \cdot \mathbb{F}^2\text{-cutrk}_{\tilde{G}}(X)$. Let $M_1 := M_G[X, V_G \setminus X]$ and $M_2 := M_G^T[V_G \setminus X, X]$. We recall that each entry of $M_{\tilde{G}}$ is of the form $a \cdot \gamma + b \cdot \tau$ for a unique pair $(a, b) \in \mathbb{F} \times \mathbb{F}$. Let π_1, π_2 and π_3 be mappings from \mathbb{F}^2 to \mathbb{F} such that:

$$\begin{aligned} \pi_1(a \cdot \gamma + b \cdot \tau) &= a, \\ \pi_2(a \cdot \gamma + b \cdot \tau) &= b. \end{aligned}$$

Clearly, $M_1 = \pi_1(M_{\tilde{G}}[X, V_G \setminus X])$ and $M_2 = \pi_2(M_{\tilde{G}}[V_G \setminus X, X])$. It is also straightforward to verify that π_1 and π_2 are homomorphism with respect to the addition. Moreover, for every $c \in \mathbb{F}$, $\delta \in \mathbb{F}^2$ and $i \in \{1, 2\}$, $\pi_i(c \cdot \delta) = c \cdot \pi_i(\delta)$.

We let v_1, \dots, v_k be a column-basis of $M_{\tilde{G}}[X, V_G \setminus X]$. Then for each column-vector v in $M_{\tilde{G}}[X, V_G \setminus X]$, $v = \sum_{i \leq k} \alpha_i \cdot v_i$ where $\alpha_i \in \mathbb{F}^2$. Then we have for $j \in \{1, 2\}$,

$$\begin{aligned} \pi_j(v) &= \sum_{i \leq k} \pi_j(\alpha_i \cdot v_i) \\ &= \sum_{i \leq k} \pi_j(\alpha_i \cdot (\pi_1(v_i) \cdot \gamma + \pi_2(v_i) \cdot \tau)) \\ &= \sum_{i \leq k} \pi_j(\alpha_i \cdot \gamma \cdot \pi_1(v_i) + \alpha_i \cdot \tau \cdot \pi_2(v_i)) \\ &= \sum_{i \leq k} \pi_j(\alpha_i \cdot \gamma) \cdot \pi_1(v_i) + \pi_j(\alpha_i \cdot \tau) \cdot \pi_2(v_i) \end{aligned}$$

Thus, every column-vector of M_j is a linear combination of $2k$ vectors $\pi_1(v_i)$ and $\pi_2(v_i)$ for $i \in \{1, \dots, k\}$, i.e. $\text{rk}(M_j) \leq 2k$. Therefore, $\mathbb{F}\text{-bicutr}k_G(X) = \text{rk}(M_1) + \text{rk}(M_2) \leq 4 \cdot \mathbb{F}^2\text{-cutrk}_{\tilde{G}}(X)$.

(2) Assume now that G is σ -symmetric. By definition of \mathbb{F} -bicutrk $_G$, we have \mathbb{F} -bicutrk $_G(X) = \text{rk}(M_G[X, V_G \setminus X]) + \text{rk}(M_G[V_G \setminus X, X])$. But since G is σ -symmetric, by Lemma 3.8, we have $\text{rk}(M_G[X, V_G \setminus X]) = \text{rk}(M_G[V_G \setminus X, X])$. We can then conclude that \mathbb{F} -bicutrk $_G(X) = 2 \cdot \mathbb{F}$ -cutrk $_G(X)$. \square

The notion of local complementation defined in Section 3.2 also preserves the \mathbb{F} -bi-rank-width.

Lemma 4.8 *Let G be a \mathbb{F}^* -graph and λ an element in \mathbb{F}^* . If $G * (x, \lambda)$ is the λ -local complementation of G at x , then for every subset X of V_G , we have \mathbb{F} -bicutrk $_{G*(x, \lambda)}(X) = \mathbb{F}$ -bicutrk $_G(X)$.*

Proof. Assume by symmetry that x is in X . Let y be a neighbor of x in X . If we apply a λ -local complementation at x , we obtain $M_{G*(x, \lambda)}[y, V_G \setminus X]$ by adding $\lambda \cdot M_G[y, x] \cdot M_G[x, V_G \setminus X]$ to $M_G[y, V_G \setminus X]$. Therefore, $\text{rk}(M_{G*(x, \lambda)}[X, V_G \setminus X]) = \text{rk}(M_G[X, V_G \setminus X])$. Similarly, we obtain $M_{G*(x, \lambda)}[V_G \setminus X, y]$ by adding to $M_G[V_G \setminus X, y]$ the column $\lambda \cdot M_G[V_G \setminus X, x] \cdot M_G[x, y]$. Again, $\text{rk}(M_{G*(x, \lambda)}[V_G \setminus X, X]) = \text{rk}(M_G[V_G \setminus X, X])$. We can thus conclude that \mathbb{F} -bicutrk $_{G*(x, \lambda)}(X) = \mathbb{F}$ -bicutrk $_G(X)$. \square

As a corollary, we get the following.

Corollary 4.9 *Let G and H be two \mathbb{F}^* -graphs. If H is locally equivalent to G , then the \mathbb{F} -bi-rank-width of H is equal to the \mathbb{F} -bi-rank-width of G . If H is a vertex-minor of G , then the \mathbb{F} -bi-rank-width of H is at most the \mathbb{F} -bi-rank-width of G .*

Note that the pivot-complementation in Section 3.2 is not well defined in the case of no-sigma-symmetric graphs. Currently, we do not have a characterisation of \mathbb{F}^* -graphs of bounded \mathbb{F} -bi-rank-width as the one in Theorem 3.22. We leave it as an open question. Moreover, this notion of vertex-minor is not a well-quasi-order on \mathbb{F}^* -graphs of bounded \mathbb{F} -bi-rank-width (see Remark 4.13).

4.2 Recognizing \mathbb{F} -Bi-Rank-Width at Most k

We will give here, for fixed k and a fixed finite field \mathbb{F} , a cubic-time algorithm that decides whether a \mathbb{F}^* -graph has \mathbb{F} -bi-rank-width at most k . The algorithm is in the same spirit as the one in Section 3.3.

We recall that if G is a \mathbb{F}^* -graph, we denote by $(\mathcal{M}_G, \Pi(G))$ the partitioned matroid represented over \mathbb{F} where \mathcal{M}_G is the matroid represented by the

$(V_G, V_G \cup V'_G)$ -matrix over \mathbb{F} (V'_G is an isomorphic copy of V_G)

$$V_G \begin{pmatrix} V_G & V'_G \\ I_{|V_G|} & M_G \end{pmatrix}$$

and $\Pi(G) := \{P_x \mid x \in V_G\}$ with $P_x := \{x, x'\}$.

As corollaries of Proposition 3.34 we get the followings.

Corollary 4.10 *Let G be a \mathbb{F}^* -graph. For every $X \subseteq V_G$, $\lambda_{\mathcal{M}_G}^{\Pi(G)}(P) = \mathbb{F}\text{-bicutrk}_G(X) + 1$ where $P := \{P_x \mid x \in X\}$.*

Corollary 4.11 *Let G be a \mathbb{F}^* -graph and let $p : V_G \rightarrow \Pi(G)$ be the bijective function such that $p(x) = P_x$. If (T, \mathcal{L}) is a layout of $\Pi(G)$ of $\lambda_{\mathcal{M}_G}^{\Pi(G)}$ -width $k + 1$, then $(T, \mathcal{L} \circ p)$ is a layout of V_G of $\mathbb{F}\text{-bicutrk}_G$ -width k . Conversely, if (T, \mathcal{L}) is a layout of V_G of $\mathbb{F}\text{-bicutrk}_G$ -width k , then $(T, \mathcal{L} \circ p^{-1})$ is a layout of $\Pi(G)$ of $\lambda_{\mathcal{M}_G}^{\Pi(G)}$ -width $k + 1$.*

Theorem 4.12 (Checking \mathbb{F} -Bi-Rank-Width at most k) *For a fixed finite field \mathbb{F} and a fixed integer k , there exists a cubic-time algorithm that, for a \mathbb{F}^* -graph G , either outputs a layout of V_G of $\mathbb{F}\text{-bicutrk}_G$ -width at most k or confirms that the \mathbb{F} -bi-rank-width of G is larger than k .*

Proof. Let k be fixed and let \mathcal{A} be the algorithm constructed in Theorem 3.33 for $k+1$. Let G be a \mathbb{F}^* -graph. We run the algorithm \mathcal{A} with input $(\mathcal{M}_G, \Pi(G))$. If it confirms that $\text{bwd}(\mathcal{M}_G, \Pi(G)) > k + 1$, then the \mathbb{F} -bi-rank-width of G is greater than k (Corollary 4.10). If it outputs a layout of $\Pi(G)$ of $\lambda_{\mathcal{M}_G}^{\Pi(G)}$ -width at most $k + 1$, we can transform it into a layout of V_G of $\mathbb{F}\text{-bicutrk}_G$ -width at most k by Corollary 4.11. The fact that the algorithm \mathcal{A} runs in cubic-time concludes the proof. \square

4.3 A Specialisation to Graphs

We now define our second notion of rank-width for directed graphs. We recall that a graph G is seen, also denoted by G , as the \mathbb{F}_2^* -graph where $M_G[x, y] := 1$ if and only if $(x, y) \in E_G$. The *bi-rank-width* of a graph G , denoted by $\text{brwd}(G)$, is its \mathbb{F}_2 -bi-rank-width. It is straightforward to verify that if G is undirected, i.e., if E_G is symmetric, then $\text{brwd}(G) = 2 \cdot \text{rwd}(G)$.

A directed graph is *strongly connected* if for every pair (x, y) of vertices, there is a directed path from x to y . Clearly in a strongly connected graph G , for every $\emptyset \subsetneq X \subsetneq V_G$, we have $\mathbb{F}\text{-bicutrk}_G(X) \geq 2$. It is straightforward to verify that

strongly connected graphs of bi-rank-width 2 are exactly the graphs completely decomposable by Cunningham's split decomposition of directed graphs [14].

The 1-local complementation of a directed graph seen as a \mathbb{F}_2^* -graph is the one defined by Bouchet [2] and Fon-Der-Flaass [20]. One easily verifies that if H is obtained by applying a 1-local complementation at x to G , then $(z, t) \in E_H$ if and only if:

- $(z, t) \notin E_G$, $(z, x) \in E_G$ and $(x, t) \in E_G$ or,
- $(z, t) \in E_G$, and either $(z, x) \notin E_G$ or $(x, t) \notin E_G$.

Figure 6 illustrates a 1-local complementation of a directed graph seen as a \mathbb{F}_2^* -graph.

We notice that the 1-local complementation of a directed graph seen as a \mathbb{F}_2^* -graph can be different from the one when we consider it as a σ_4 -symmetric \mathbb{F}_4^* -graph (see Section 3.4). Figures 7 and 8 illustrate this observation. We leave open the question of finding a notion of vertex-minor for directed graphs, that not only lets invariant \mathbb{F}_4 -rank-width and \mathbb{F}_2 -bi-rank-width, but also is independent of the representation.

Remark 4.13 *Directed graphs of bounded bi-rank-width are not well-quasi-ordered by the vertex-minor relation. In fact the class \mathcal{EC} of directed even cycles such that each vertex has either in-degree 2 or out-degree 2 are of bounded bi-rank-width and are not well-quasi-ordered by vertex-minor relation since none of them is a vertex-minor of another. In fact each of them is isomorphic to its 1-local complementations. Figure 7 illustrates such cycles.*

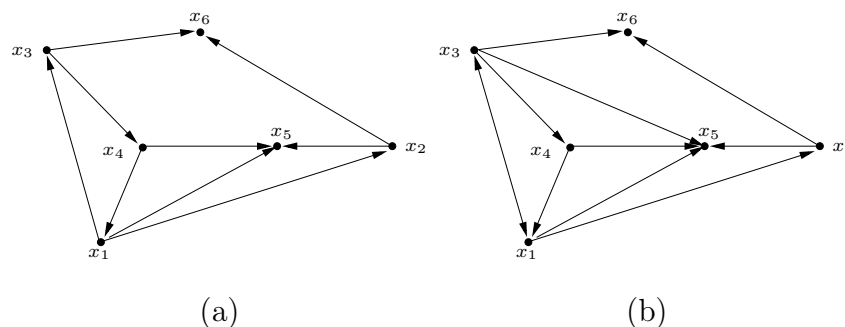


Figure 6. (a) A directed graph seen as a \mathbb{F}_2^* -graph. (b) Its 1-local complementation at x_4 .

5 Algebraic Operations for \mathbb{F} -Rank-Width and \mathbb{F} -Bi-Rank-Width

Courcelle and the first author gave in [9] graph operations that characterise exactly the notion of rank-width of undirected graphs. These operations are

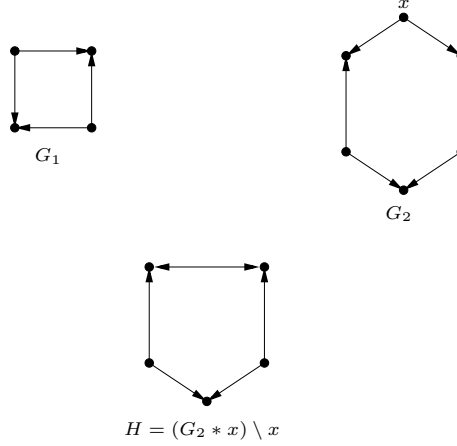


Figure 7. G_1 and G_2 are graphs in \mathcal{EC} . Each graph in \mathcal{EC} , seen as a \mathbb{F}_2^* -graph, is isomorphic to its 1-local complementations. This is not the case if we consider them as a σ_4 -symmetric \mathbb{F}_4^* -graph. For instance, H is a vertex-minor of G_2 seen as a σ_4 -symmetric \mathbb{F}_4^* -graph.

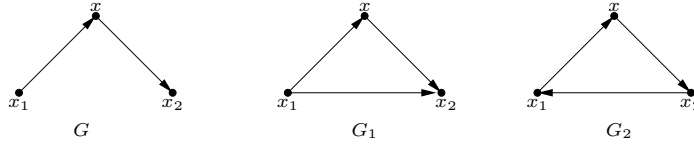


Figure 8. G_1 is a vertex-minor of G seen as a \mathbb{F}_2^* -graph and the only one locally equivalent to it, and G_2 is a vertex-minor of G seen as a σ_4 -symmetric \mathbb{F}_4^* -graph. G_1 is not isomorphic to G_2 .

interesting because they allow to check monadic second-order properties on undirected graph classes of bounded rank-width without using clique-width operations. This important in a practical point of view since it allows to decrease by 1 the hidden towers of exponents due to the generality of the method. We give in Section 5.1 graph operations, that generalise the ones in [9] and that characterise exactly \mathbb{F} -rank-width. A specialisation that allows to characterise exactly \mathbb{F} -bi-rank-width is then presented in Section 5.2.

We let \mathbb{F} be a fixed finite field along this section. For a fixed positive integer k , we let \mathbb{F}^k be the set of row vectors of length k .

5.1 Operations Characterising \mathbb{F} -Rank-Width

The operations are easy adaptations of the ones in [9]. We let $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ be a fixed sesqui-morphism. If $u := (u_1, \dots, u_k) \in \mathbb{F}^k$, we let $\sigma(u)$ be $(\sigma(u_1), \dots, \sigma(u_k))$. Similarly, if $M = (m_{i,j})$ is a matrix, we let $\sigma(M)$ be the matrix $(\sigma(m_{i,j}))$. In this section we deal with σ -symmetric \mathbb{F}^* -graphs. So, we will say graph instead of σ -symmetric \mathbb{F}^* -graph.

An \mathbb{F}^k -coloring of a graph G is a mapping $\gamma_G : V_G \rightarrow \mathbb{F}^k$ with no constraint on the values of γ for adjacent vertices. An \mathbb{F}^k -colored graph G is a tuple $(V_G, E_G, \ell_G, \gamma_G)$ where (V_G, E_G, ℓ_G) is a graph and γ_G is an \mathbb{F}^k -coloring of (V_G, E_G, ℓ_G) . Notice that an \mathbb{F}^k -colored graph has not only its edges colored with colors from \mathbb{F} , but also its vertices with colors from \mathbb{F}^k . With an \mathbb{F}^k -colored graph G , we associate the $(V_G \times [k])$ -matrix Γ_G , the row vectors of which are the vectors $\gamma_G(x)$ in \mathbb{F}^k for x in V_G .

The following is a binary graph operation that combines several operations consisting in adding colored edges between its disjoint arguments and recolor them independently.

Definition 5.1 (Bilinear Products) *Let k, ℓ and m be positive integers and let M, N and P be $k \times \ell$, $k \times m$ and $\ell \times m$ matrices, respectively, over \mathbb{F} . For an \mathbb{F}^k -colored graph G and an \mathbb{F}^ℓ -colored graph H , we let $G \otimes_{M,N,P} H$ be the \mathbb{F}^m -colored graph $K := (V_G \cup V_H, E_G \cup E_H \cup E', \ell_K, \gamma_K)$ where:*

$$\begin{aligned} E' &:= \{xy \mid x \in V_G, y \in V_H \text{ and } \gamma_G(x) \cdot M \cdot \sigma(\gamma_H(y))^T \neq 0\}, \\ \ell_K((x, y)) &:= \begin{cases} \ell_G((x, y)) & \text{if } x, y \in V_G, \\ \ell_H((x, y)) & \text{if } x, y \in V_H, \\ \gamma_G(x) \cdot M \cdot \sigma(\gamma_H(y))^T & \text{if } x \in V_G, y \in V_H, \\ \sigma(\gamma_G(y) \cdot M \cdot \sigma(\gamma_H(x))^T) & \text{if } y \in V_G, x \in V_H. \end{cases} \\ \gamma_K(x) &:= \begin{cases} \gamma_G(x) \cdot N & \text{if } x \in V_G, \\ \gamma_H(x) \cdot P & \text{if } x \in V_H. \end{cases} \end{aligned}$$

Definition 5.2 (Constants) *For each $u \in \mathbb{F}^k$, we let \mathbf{u} be a constant denoting a \mathbb{F}^k -colored graph with one vertex colored by u and no edge.*

We denote by $\mathcal{C}_n^{\mathbb{F}}$ the set $\{\mathbf{u} \mid u \in \mathbb{F}^1 \cup \dots \cup \mathbb{F}^n\}$. We let $\mathcal{R}_n^{(\mathbb{F}, \sigma)}$ be the set of bilinear products $\otimes_{M,N,P}$ where M, N and P are respectively $k \times \ell$, $k \times m$ and $\ell \times m$ matrices for $k, \ell, m \leq n$. Each term t in $T(\mathcal{R}_n^{(\mathbb{F}, \sigma)}, \mathcal{C}_n^{\mathbb{F}})$ defines, up to isomorphism, a σ -symmetric \mathbb{F}^* -graph $val(t)$. We write by abuse of notation $G = val(t)$ to say that G is isomorphic to $val(t)$.

One easily verifies that the operations $\otimes_{M,N,P}$ can be defined in terms of the disjoint union and quantifier-free operations. The following is thus a corollary of results in [5,10].

Theorem 5.3 *For each monadic second-order property φ , there exists an algorithm that checks for every term $t \in T(\mathcal{R}_n^{(\mathbb{F}, \sigma)}, \mathcal{C}_n^{\mathbb{F}})$, in time $O(|t|)$, if the σ -symmetric \mathbb{F}^* -graph defined by this term, up to isomorphism, satisfies φ .*

The principal result of this section is the following.

Theorem 5.4 *A graph G has \mathbb{F} -rank-width at most n if and only if it is*

isomorphic to $\text{val}(t)$ for some term t in $T(\mathcal{R}_n^{(\mathbb{F}, \sigma)}, \mathcal{C}_n^{\mathbb{F}})$.

Let $\sigma_1 : \mathbb{F}_2 \rightarrow \mathbb{F}_2$ be the identity automorphism. As a corollary of Theorem 5.4, we get the following.

Theorem 5.5 ([9]) *An undirected graph has rank-width at most n if and only if it is isomorphic to $\text{val}(t)$ for some term t in $T(\mathcal{R}_n^{(\mathbb{F}_2, \sigma_1)}, \mathcal{C}_n^{\mathbb{F}_2})$.*

We can now begin the proof of Theorem 5.4. It is similar to the one in [9].

Lemma 5.6 *If $K = G \otimes_{M,N,P} H$, then $M_K[V_G, V_H] = \Gamma_G \cdot M \cdot \sigma(\Gamma_H)^T$ and $\Gamma_K = \begin{pmatrix} \Gamma_G \cdot N \\ \Gamma_H \cdot P \end{pmatrix}$. Moreover, K is isomorphic to $H \otimes_{M',P,N} G$ where $M' = \frac{1}{\sigma(1)^2} \cdot \sigma(M)^T$.*

Lemma 5.7 *Let $t = c \bullet t'$ where $t' \in T(\mathcal{R}_n^{(\mathbb{F}, \sigma)}, \mathcal{C}_n^{\mathbb{F}})$ and $c \in \text{Cxt}(\mathcal{R}_n^{(\mathbb{F}, \sigma)}, \mathcal{C}_n^{\mathbb{F}}) \setminus \text{Id}$. If $G = \text{val}(t)$ and $H = \text{val}(t')$, then*

$$\begin{aligned} M_G[V_H, V_G \setminus V_H] &= \Gamma_H \cdot B, \\ \Gamma_{G[V_H]} &= \Gamma_H \cdot C. \end{aligned}$$

for some matrices B and C .

Proof. We prove it by induction on the structure of c . We identify two cases (the two other cases are similar by symmetry and Lemma 5.6).

Case 1 $c = \text{id} \otimes_{M,N,P} t''$. Then, $G = H \otimes_{M,N,P} K$ where $K = \text{val}(t'')$. By Lemma 5.6,

$$\begin{aligned} M_G[V_H, V_G \setminus V_H] &= \Gamma_H \cdot M \cdot \sigma(\Gamma_K)^T, \\ \Gamma_{G[H]} &= \Gamma_H \cdot N. \end{aligned}$$

We let $B = M \cdot \sigma(\Gamma_K)^T$ and $C = N$.

Case 2 $c = c' \otimes_{M,N,P} t''$. We let $G' = \text{val}(c' \bullet t')$ and $K = \text{val}(t'')$. Hence, $G = G' \otimes_{M,N,P} K$. By definition and Lemma 5.6,

$$M_G[V_H, V_G \setminus V_H] = \begin{pmatrix} M_{G'}[V_H, V_{G'} \setminus V_H] & (\Gamma_{G'} \cdot M \cdot \sigma(\Gamma_K)^T)[V_H, V_K] \end{pmatrix}$$

By inductive hypothesis, $M_{G'}[V_H, V_{G'} \setminus V_H] = \Gamma_H \cdot B'$ for some matrix B' . Moreover, $(\Gamma_{G'} \cdot M \cdot \sigma(\Gamma_K)^T)[V_H, V_K] = \Gamma_{G'[V_H]} \cdot M \cdot \sigma(\Gamma_K)^T$. But by inductive hypothesis, $\Gamma_{G'[V_H]} = \Gamma_H \cdot C'$ for some matrix C' . Then, $M_G[V_H, V_G \setminus V_H] = \Gamma_H \cdot B$ where $B = \begin{pmatrix} B' & C' \cdot M \cdot \sigma(\Gamma_K)^T \end{pmatrix}$. Moreover, $\Gamma_{G[H]} = \Gamma_H \cdot C$ where $C = C' \cdot N$ since $\Gamma_{G[V_H]} = \Gamma_{G'[V_H]} \cdot N$. \square

We now prove the “if direction” of Theorem 5.4 in the following.

Proposition 5.8 *If G is isomorphic to $\text{val}(t)$ for a term t in $T(\mathcal{R}_n^{(\mathbb{F}, \sigma)}, \mathcal{C}_n^{\mathbb{F}})$, then $\mathbb{F}\text{-rd}(G) \leq n$.*

Proof. Let T be the syntactic tree of t . By definition, there exists a bijective function $\mathcal{L} : V_G \rightarrow L_T$ where L_T is the set of leaves of t , then of T . We let (T, \mathcal{L}) be a layout of V_G . In order to prove that the \mathbb{F} -cutrk $_G$ -width of (T, \mathcal{L}) is at most n , it is sufficient to prove that for each subgraph H of G associated to a sub-term t' of t , $\mathbb{F}\text{-cutrk}_G(V_H) \leq n$. However, we have proved in Lemma 5.7 that $M_G[V_H, V_G \setminus V_H] = \Gamma_H \cdot B$ for some matrix B . And since each such H is \mathbb{F}^k -colored for some $k \leq n$, we are done. \square

The following proves the “only if direction” of Theorem 5.4.

Proposition 5.9 *If $\mathbb{F}\text{-rd}(G) \leq n$, then G is isomorphic to $\text{val}(t)$ for a term t in $T(\mathcal{R}_n^{(\mathbb{F}, \sigma)}, \mathcal{C}_n^{\mathbb{F}})$.*

Let us first introduce another notion. Let V be a subset of V_G . A subset X of V is called a *vertex-basis* of $M_G[V, V_G \setminus V]$ if $\{M_G[x, V_G \setminus V] \mid x \in X\}$ is linearly independent and generates the row space of $M_G[V, V_G \setminus V]$.

Proof. Assume first that G is connected. Let (T, \mathcal{L}) be a layout of V_G of \mathbb{F} -cutrk $_G$ -width at most n . We pick an edge of T , subdivide it and root the new tree T' by considering the new node as the root. For each node u of T' , we let G_u be the subgraph of G induced by the vertices that are in correspondence with the leaves of the sub-tree of T' rooted at u . We let $r(u)$ be $\mathbb{F}\text{-cutrk}_G(V_{G_u})$.

Lemma 5.10 *For each node u of T' , we can construct a term t_u in $T(\mathcal{R}_n^{(\mathbb{F}, \sigma)}, \mathcal{C}_n^{\mathbb{F}})$ such that $\text{val}(t_u)$ is isomorphic to G_u and is a $\mathbb{F}^{r(u)}$ -colored graph. There exists moreover a vertex-basis X_u of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ such that $M_G[V_{G_u}, V_G \setminus V_{G_u}] = \Gamma_{\text{val}(t_u)} \cdot M_G[X_u, V_G \setminus V_{G_u}]$.*

It is clear that if r is the root of T' , then $G = \text{val}(t_r)$ where t_r is the term in $T(\mathcal{R}_n^{(\mathbb{F}, \sigma)}, \mathcal{C}_n^{\mathbb{F}})$ constructed in Lemma 5.10.

Assume now that G is not connected and let G_1, \dots, G_m be the connected components of G . By Lemma 5.10, we can construct terms t_1, \dots, t_m that defines, up to isomorphism, respectively, G_1, \dots, G_m . It is clear that $((\dots((t_1 \otimes_{O,O,O} t_2) \otimes_{O,O,O} t_3) \dots) \otimes_{O,O,O} t_m)$ is isomorphic to G where O is the null matrix of order 1×1 . This concludes the proof of the proposition and therefore of Theorem 5.4. \square

Proof of Lemma 5.10. We prove it by induction on the number of vertices

of G_u . Let u be a node in T' .

If G_u is a single vertex x , then since G is connected we let $t_u := \mathbf{1}$ and $X_u := \{x\}$. It is clear that t_u and X_u verify the statements of the lemma.

Assume now that G_u has at least two vertices. Then u has two sons u_1 and u_2 so that G_{u_1} and G_{u_2} have less vertices than G_u . By inductive hypothesis, there exist t_{u_i} and X_{u_i} , for $i = 1, 2$, verifying the statements of the lemma. We let $r(u_1) := h$ and $r(u_2) := k$. We let $X_{u_1} := \{x_1, \dots, x_h\}$ and $X_{u_2} := \{y_1, \dots, y_k\}$. We let $M := \frac{1}{\sigma(1)} \cdot M_G[X_{u_1}, X_{u_2}]$, and $H = \text{val}(t_{u_1})$ and $K = \text{val}(t_{u_2})$.

Claim 5.11 $M_G[V_{G_{u_1}}, V_{G_{u_2}}] = \Gamma_H \cdot M \cdot \sigma(\Gamma_K)^T$.

Proof of Claim 5.11. Let $x \in V_{G_{u_1}}$ and $y \in V_{G_{u_2}}$. By inductive hypothesis, $M_G[x, V_G \setminus V_{G_{u_1}}] = \gamma_H(x) \cdot M_G[X_{u_1}, V_G \setminus V_{G_{u_1}}]$ and $M_G[y, V_G \setminus V_{G_{u_2}}] = \gamma_K(y) \cdot M_G[X_{u_2}, V_G \setminus V_{G_{u_2}}]$. Hence, $\gamma_H(x) \cdot M = \frac{1}{\sigma(1)} \cdot M_G[x, X_{u_2}]$. Therefore,

$$\begin{aligned} \gamma_H(x) \cdot M \cdot \sigma(\gamma_K(y))^T &= \frac{1}{\sigma(1)} \cdot M_G[x, X_{u_2}] \cdot \sigma(\gamma_K(y))^T \\ &= \frac{1}{\sigma(1)} \cdot \sigma(M_G[X_{u_2}, x])^T \cdot \sigma(\gamma_K(y))^T \\ &= \frac{1}{\sigma(1)} \cdot \sigma(\gamma_K(y)) \cdot \sigma(M_G[X_{u_2}, x]) \\ &= \sigma(\gamma_K(y) \cdot M_G[X_{u_2}, x]) \\ &= \sigma(M_G[y, x]) = M_G[x, y]. \quad \square \end{aligned}$$

It remains now to find a vertex-basis X_u of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ and matrices N and P such that $M_G[V_{G_u}, V_G \setminus V_{G_u}] = \begin{pmatrix} \Gamma_H \cdot N \\ \Gamma_K \cdot P \end{pmatrix} \cdot M_G[X_u, V_G \setminus V_{G_u}]$.

It is straightforward to verify that $\{M_G[z, V_G \setminus V_{G_u}] \mid z \in X_{u_1} \cup X_{u_2}\}$ generates the row space of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$. Therefore, we can find a vertex-basis X_u of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ which is a subset $X_{u_1} \cup X_{u_2}$. That means, for each $z \in X_{u_1} \cup X_{u_2}$, there exists a row vector b_z such that $M_G[z, V_G \setminus V_{G_u}] = b_z \cdot M_G[X_u, V_G \setminus V_{G_u}]$. We let $t_u = t_{u_1} \otimes_{M, N, P} t_{u_2}$ where:

$$N := \begin{pmatrix} b_{x_1} & \dots & b_{x_h} \end{pmatrix}^T \quad P := \begin{pmatrix} b_{y_1} & \dots & b_{y_k} \end{pmatrix}^T$$

From Claim 5.11 it remains to show that $\Gamma_H \cdot N \cdot M_G[X_u, V_G \setminus V_{G_u}] = M_G[V_{G_{u_1}}, V_G \setminus V_{G_u}]$ and $\Gamma_K \cdot P \cdot M_G[X_u, V_G \setminus V_{G_u}] = M_G[V_{G_{u_2}}, V_G \setminus V_{G_u}]$. For that it is sufficient to prove, for each t in $V_G \setminus V_{G_u}$, that $M_G[X_{u_1}, t] = N \cdot M_G[X_u, t]$ and $M_G[X_{u_2}, t] = P \cdot M_G[X_u, t]$. But, this is a straightforward computation by the definitions of N , P and X_u . This concludes the proof of the lemma. \square

5.2 Graph Operations for \mathbb{F} -Bi-Rank-Width

In this section we specialise the graph operations in $\mathcal{R}_n^{(\mathbb{F}, \sigma)}$ in order to give graph operations that characterise exactly \mathbb{F} -bi-rank-width. We start by some notations. Let k_1 and k_2 be positive integers. An \mathbb{F}^{k_1, k_2} -bi coloring of a \mathbb{F}^* -graph G is a couple of mappings $\gamma_G^+ : V_G \rightarrow \mathbb{F}^{k_1}$ and $\gamma_G^- : V_G \rightarrow \mathbb{F}^{k_2}$. An \mathbb{F}^{k_1, k_2} -bi colored graph is a tuple $(V_G, E_G, \ell_G, \gamma_G^+, \gamma_G^-)$ where (V_G, E_G, ℓ_G) is a \mathbb{F}^* -graph and (γ_G^+, γ_G^-) is a \mathbb{F}^{k_1, k_2} -bi coloring. With an \mathbb{F}^{k_1, k_2} -bi colored graph G we associate the $(V_G, [k_1])$ and $(V_G, [k_2])$ -matrices Γ_G^+ and Γ_G^- , the row vectors of which are respectively $\gamma_G^+(x)$ and $\gamma_G^-(x)$ for x in V_G .

Definition 5.12 *Let $k_1, k_2, \ell_1, \ell_2, m_1$ and m_2 be positive integers. Let M_1, M_2, N_1, N_2, P_1 and P_2 be respectively $k_1 \times \ell_1, k_2 \times \ell_2, k_1 \times m_1, k_2 \times m_2, \ell_1 \times m_1$ and $\ell_2 \times m_2$ -matrices. For a \mathbb{F}^{k_1, k_2} -bi colored graph G and a $\mathbb{F}^{\ell_1, \ell_2}$ -bi colored graph H , we let $G \otimes_{M_1, M_2, N_1, N_2, P_1, P_2} H$ be the \mathbb{F}^{m_1, m_2} -bi colored graph $K := (V_G \cup V_H, E_G \cup E_H \cup E_1 \cup E_2, \ell_K, \gamma_K^+, \gamma_K^-)$ where:*

$$\begin{aligned} E_1 &:= \{(x, y) \mid x \in V_G, y \in V_H \text{ and } \gamma_G^+(x) \cdot M_1 \cdot (\gamma_H^-(y))^T \neq 0\}, \\ E_2 &:= \{(y, x) \mid x \in V_G, y \in V_H \text{ and } \gamma_G^-(x) \cdot M_2 \cdot (\gamma_H^+(y))^T \neq 0\}, \\ \ell_K((x, y)) &:= \begin{cases} \ell_G((x, y)) & \text{if } x, y \in V_G, \\ \ell_H((x, y)) & \text{if } x, y \in V_H, \\ \gamma_G^+(x) \cdot M_1 \cdot (\gamma_H^-(y))^T & \text{if } x \in V_G \text{ and } y \in V_H, \\ \gamma_G^-(y) \cdot M_2 \cdot (\gamma_H^+(x))^T & \text{if } y \in V_G \text{ and } x \in V_H, \end{cases} \\ \gamma_K^+(x) &:= \begin{cases} \gamma_G^+(x) \cdot N_1 & \text{if } x \in V_G, \\ \gamma_H^+(x) \cdot P_1 & \text{if } x \in V_H, \end{cases} \\ \gamma_K^-(x) &:= \begin{cases} \gamma_G^-(x) \cdot N_2 & \text{if } x \in V_G, \\ \gamma_H^-(x) \cdot P_2 & \text{if } x \in V_H, \end{cases} \end{aligned}$$

Definition 5.13 *For each pair $(u, v) \in \mathbb{F}^{k_1} \times \mathbb{F}^{k_2}$, we let $\mathbf{u} \cdot \mathbf{v}$ be the constant denoting a \mathbb{F}^{k_1, k_2} -bi colored graph with a single vertex and no edge.*

We let $\mathcal{BC}_n^{\mathbb{F}}$ be the set $\{\mathbf{u} \cdot \mathbf{v} \mid (u, v) \in \mathbb{F}^{k_1} \times \mathbb{F}^{k_2} \text{ and } k_1 + k_2 \leq n\}$. We denote by $\mathcal{BR}_n^{\mathbb{F}}$ the set of all operations $\otimes_{M_1, M_2, N_1, N_2, P_1, P_2}$ where M_1, M_2, N_1, N_2, P_1 and P_2 are respectively $k_1 \times \ell_1, k_2 \times \ell_2, k_1 \times m_1, k_2 \times m_2, \ell_1 \times m_1$ and $\ell_2 \times m_2$ -matrices and $k_1 + k_2, \ell_1 + \ell_2$ and $m_1 + m_2 \leq n$. Every term t in $T(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}})$ defines, up to isomorphism, a \mathbb{F}^* -graph denoted by $\text{val}(t)$.

The operations in $\mathcal{BR}_n^{\mathbb{F}}$ can be defined in terms of disjoint union and quantifier-free operations. Therefore, Theorem 5.3 is still true if we replace $\mathcal{R}_n^{(\mathbb{F}, \sigma)}$ by $\mathcal{BR}_n^{\mathbb{F}}$. The principal result of this section is the following.

Theorem 5.14 *A \mathbb{F}^* -graph has \mathbb{F} -bi-rank-width at most n if and only if it is*

isomorphic to some term t in $T(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}})$.

The proof is similar to the one of Theorem 5.4. The following lemma is straightforward to verify.

Lemma 5.15 *If $K = G \otimes_{M_1, M_2, N_1, N_2, P_1, P_2} H$, then*

$$\begin{aligned} M_K[V_G, V_H] &= \Gamma_G^+ \cdot M_1 \cdot (\Gamma_H^-)^T, & M_K[V_H, V_G] &= (\Gamma_G^- \cdot M_2 \cdot (\Gamma_H^+)^T)^T, \\ \Gamma_K^+ &= \begin{pmatrix} \Gamma_G^+ \cdot N_1 \\ \Gamma_H^+ \cdot P_1 \end{pmatrix}, & \Gamma_K^- &= \begin{pmatrix} \Gamma_G^- \cdot N_2 \\ \Gamma_H^- \cdot P_2 \end{pmatrix}. \end{aligned}$$

Moreover, K is isomorphic to $H \otimes_{(M_2)^T, (M_1)^T, P_1, P_2, N_1, N_2} G$.

Lemma 5.16 *Let $t = \mathbf{c} \bullet t'$ where $t' \in T(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}})$ and $c \in \text{Cxt}(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}}) \setminus \text{Id}$. If $G = \text{val}(t)$ and $H = \text{val}(t')$, then $M_G[V_H, V_G \setminus V_H] = \Gamma_H^+ \cdot B_1$ and $M_G[V_G \setminus V_H, V_H] = (\Gamma_H^- \cdot B_2)^T$ for some matrices B_1 and B_2 .*

Proof. We prove it by induction on the structure of c , by showing in addition that $\Gamma_{G[V_H]}^+ = \Gamma_H^+ \cdot C_1$ and $\Gamma_{G[V_H]}^- = \Gamma_H^- \cdot C_2$ for some matrices C_1 and C_2 . We identify two cases (the two other cases are similar by symmetry and Lemma 5.15).

Case 1 $c = \text{Id} \otimes_{M_1, M_2, N_1, N_2, P_1, P_2} t''$. We let $K = \text{val}(t'')$. Then $G = H \otimes_{M_1, M_2, N_1, N_2, P_1, P_2} K$. By Lemma 5.15,

$$\begin{aligned} M_G[V_H, V_G \setminus V_H] &= \Gamma_H^+ \cdot M_1 \cdot (\Gamma_K^-)^T, & M_G[V_G \setminus V_H, V_H] &= (\Gamma_H^- \cdot M_2 \cdot (\Gamma_K^+)^T)^T, \\ \Gamma_{G[V_H]}^+ &= \Gamma_H^+ \cdot N_1, & \Gamma_{G[V_H]}^- &= \Gamma_H^- \cdot N_2. \end{aligned}$$

We let $B_1 = M_1 \cdot (\Gamma_K^-)^T$, $B_2 = M_2 \cdot (\Gamma_K^+)^T$, $C_1 = N_1$ and $C_2 = N_2$.

Case 2 $c = c' \otimes_{M, M', N, P} t''$ where $c' \in \text{Cxt}(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}}) \setminus \text{Id}$. We let $K = \text{val}(t'')$ and $G' = \text{val}(c' \bullet t')$. Hence $G = G' \otimes_{M_1, M_2, N_1, N_2, P_1, P_2} K$. By Lemma 5.15,

$$\begin{aligned} M_G[V_H, V_G \setminus V_H] &= \begin{pmatrix} M_{G'}[V_H, V_{G'} \setminus V_H] & \Gamma_{G'[V_H]}^+ \cdot M_1 \cdot (\Gamma_K^-)^T \end{pmatrix}, \\ M_G[V_G \setminus V_H, V_H] &= \begin{pmatrix} M_{G'}[V_{G'} \setminus V_H, V_H] & (\Gamma_{G'[V_H]}^- \cdot M_2 \cdot (\Gamma_K^+)^T)^T \end{pmatrix} \end{aligned}$$

By inductive hypothesis, $M_{G'}[V_H, V_{G'} \setminus V_H] = \Gamma_H^+ \cdot B'_1$ and $M_{G'}[V_{G'} \setminus V_H, V_H] = (\Gamma_H^- \cdot B'_2)^T$. Moreover, $\Gamma_{G'[V_H]}^+ = \Gamma_H^+ \cdot C'_1$ and $\Gamma_{G'[V_H]}^- = \Gamma_H^- \cdot C'_2$. Therefore, letting

$$\begin{aligned} B_1 &= \begin{pmatrix} B'_1 & C'_1 \cdot M_1 \cdot (\Gamma_K^-)^T \end{pmatrix}, & B_2 &= \begin{pmatrix} B'_2 & C'_2 \cdot M_2 \cdot (\Gamma_K^+)^T \end{pmatrix}, \\ C_1 &= C'_1 \cdot N_1, & C_2 &= C'_2 \cdot N_2 \end{aligned}$$

concludes the proof. \square

The following proves the “if direction” of Theorem 5.14.

Proposition 5.17 *If G is isomorphic to $\text{val}(t)$ for a term t in $T(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}})$, then $\mathbb{F}\text{-brwd}(G) \leq n$.*

Proof. Let T be the syntactic tree of t . By definition, there exists a bijective function $\mathcal{L} : V_G \rightarrow L_T$ where L_T is the set of leaves of t , then of T . We let (T, \mathcal{L}) be a layout of V_G . In order to prove that the \mathbb{F} -bicutr $_G$ -width of (T, \mathcal{L}) is at most n , it is sufficient to prove that for each subgraph H of G associated to a sub-term t' of t , $\mathbb{F}\text{-bicutr}_G(V_H) \leq n$. However, we have proved in Lemma 5.16 that $M_G[V_H, V_G \setminus V_H] = \Gamma_H^+ \cdot B_1$ and $M_G[V_G \setminus V_H, V_H] = (\Gamma_H^- \cdot B_2)^T$ for some matrices B_1 and B_2 . And since each such H is \mathbb{F}^{k_1, k_2} -colored where $k_1 + k_2 \leq n$, we are done. \square

The following proves the “only if direction” of Theorem 5.14.

Proposition 5.18 *If $\mathbb{F}\text{-brwd}(G) \leq n$, then G is isomorphic to $\text{val}(t)$ for a term t in $T(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}})$.*

Proof. Assume first that G is connected. Let (T, \mathcal{L}) be a layout of V_G of \mathbb{F} -bicutr $_G$ -width at most n . We pick an edge of T , subdivide it and root the new tree T' by considering the new node as the root. For each node u of T' , we let G_u be the subgraph of G induced by the vertices that are in correspondence with the leaves of the sub-tree of T' rooted at u . We let $r_1(u)$ be $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ and $r_2(u)$ be $M_G[V_G \setminus V_{G_u}, V_{G_u}]$.

Lemma 5.19 *For each node u of T' , we can construct a term t_u in $T(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}})$ such that $\text{val}(t_u)$ is isomorphic to G_u and is a $\mathbb{F}^{r_1(u), r_2(u)}$ -bi colored graph. There exists moreover vertex-bases X_u^+ and X_u^- of, respectively, $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ and $(M_G[V_G \setminus V_{G_u}, V_{G_u}])^T$ such that $M_G[V_{G_u}, V_G \setminus V_{G_u}] = \Gamma_{\text{val}(t_u)}^+ \cdot M_G[X_u^+, V_G \setminus V_{G_u}]$ and $M_G[V_G \setminus V_{G_u}, V_{G_u}] = M_G[V_G \setminus V_{G_u}, X_u^-] \cdot (\Gamma_{\text{val}(t_u)}^-)^T$.*

It is clear that if r is the root of T' , then $G = \text{val}(t_r)$ where t_r is the term in $T(\mathcal{BR}_n^{\mathbb{F}}, \mathcal{BC}_n^{\mathbb{F}})$ constructed in Lemma 5.19.

Assume now that G is not connected and let G_1, \dots, G_m be the connected components of G . By Lemma 5.19, we can construct terms t_1, \dots, t_m that defines, up to isomorphism, respectively, G_1, \dots, G_m . It is clear that $((\dots((t_1 \otimes_{O, O, O, O, O, O} t_2) \otimes_{O, O, O, O, O, O} t_3) \dots) \otimes_{O, O, O, O, O, O} t_m)$ is isomorphic to G where O is the null

matrix of order 1. This concludes the proof of the proposition and therefore of Theorem 5.14. \square

Proof of Lemma 5.19. We prove it by induction on the number of vertices of G_u . Let u be a node in T' .

If G_u is a single vertex x , then since G is connected we let $t_u := \mathbf{1} \cdot \mathbf{1}$ and $X_u^+ := X_u^- := \{x\}$. It is clear that t_u , X_u^+ and X_u^- verify the statements of the lemma.

Assume now that G_u has at least two vertices. Then u has two sons u_1 and u_2 so that G_{u_1} and G_{u_2} have less vertices than G_u . By inductive hypothesis, there exist t_{u_i} , $X_{u_i}^+$ and $X_{u_i}^-$, for $i = 1, 2$, verifying the statements of the lemma. We let $r_1(u_1) := h$, $r_2(u_1) = h'$, $r_1(u_2) = k$ and $r_2(u_2) := k'$. We let $X_{u_1}^+ := \{x_{i_1}, \dots, x_{i_h}\}$, $X_{u_1}^- := \{x_{j_1}, \dots, x_{j_{h'}}\}$, $X_{u_2}^+ := \{y_{t_1}, \dots, y_{t_k}\}$ and $X_{u_2}^- := \{y_{s_1}, \dots, y_{s_{k'}}\}$. We let $M_1 := M_G[X_{u_1}^+, X_{u_2}^-]$ and $M_2 := (M_G[X_{u_2}^+, X_{u_1}^-])^T$, and $H = \text{val}(t_{u_1})$ and $K = \text{val}(t_{u_2})$.

Claim 5.20 $M_G[V_{G_{u_1}}, V_{G_{u_2}}] = \Gamma_H^+ \cdot M_1 \cdot (\Gamma_K^-)^T$ and $M_G[V_{G_{u_2}}, V_{G_{u_1}}] = (\Gamma_H^- \cdot M_2 \cdot (\Gamma_K^+)^T)^T$.

Proof of Claim 5.20. Let $x \in V_{G_{u_1}}$ and $y \in V_{G_{u_2}}$. By inductive hypothesis,

$$\begin{aligned} M_G[x, V_G \setminus V_{G_{u_1}}] &= \gamma_H^+(x) \cdot M_G[X_{u_1}^+, V_G \setminus V_{G_{u_1}}], \\ M_G[V_G \setminus V_{G_{u_1}}, x] &= M_G[V_G \setminus V_{G_{u_1}}, X_{u_1}^-] \cdot (\gamma_H^-(x))^T \\ M_G[y, V_G \setminus V_{G_{u_2}}] &= \gamma_K^+(y) \cdot M_G[X_{u_2}^+, V_G \setminus V_{G_{u_2}}], \\ M_G[V_G \setminus V_{G_{u_2}}, y] &= M_G[V_G \setminus V_{G_{u_2}}, X_{u_2}^-] \cdot (\gamma_K^-(y))^T. \end{aligned}$$

Hence,

$$\gamma_H^+(x) \cdot M_1 \cdot (\gamma_K^-(y))^T = M_G[x, X_{u_2}^-] \cdot (\gamma_K^-(y))^T = M_G[x, y],$$

and

$$\begin{aligned} \gamma_H^-(x) \cdot M_2 \cdot (\gamma_K^+(y))^T &= \gamma_H^-(x) \cdot (M_G[X_{u_2}^+, X_{u_1}^-])^T \cdot (\gamma_K^+(y))^T \\ &= (M_G[X_{u_2}^+, X_{u_1}^-] \cdot (\gamma_H^-(x))^T)^T \cdot (\gamma_K^+(y))^T \\ &= (M_G[X_{u_2}^+, x])^T \cdot (\gamma_K^+(y))^T \\ &= (\gamma_K^+(y) \cdot M_G[X_{u_2}^+, x])^T = M_G[y, x]. \quad \square \end{aligned}$$

It remains now to find vertex-bases X_u^+ and X_u^- of, respectively, $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ and $(M_G[V_G \setminus V_{G_u}, V_{G_u}])^T$, and matrices N_1, N_2, P_1 and P_2 such that $M_G[V_{G_u}, V_G \setminus V_{G_u}] = \begin{pmatrix} \Gamma_H^+ \cdot N_1 \\ \Gamma_K^+ \cdot P_1 \end{pmatrix} \cdot M_G[X_u^+, V_G \setminus V_{G_u}]$ and $M_G[V_G \setminus V_{G_u}, V_{G_u}] = M_G[V_G \setminus V_{G_u}, X_u^-] \cdot \begin{pmatrix} \Gamma_H^- \cdot N_2 \\ \Gamma_K^- \cdot P_2 \end{pmatrix}^T$.

It is straightforward to verify that $\{M_G[z, V_G \setminus V_{G_u}] \mid z \in X_{u_1}^+ \cup X_{u_2}^+\}$ generates the row space of $M_G[V_{G_u}, V_G \setminus V_{G_u}]$. Similarly, $\{(M_G[V_G \setminus V_{G_u}, z])^T \mid z \in X_{u_1}^- \cup X_{u_2}^-\}$ generates the row space of $(M_G[V_G \setminus V_{G_u}, V_{G_u}])^T$. Therefore, we can find vertex-bases $X_u^+ \subseteq X_{u_1}^+ \cup X_{u_2}^+$ and $X_u^- \subseteq X_{u_1}^- \cup X_{u_2}^-$ of, respectively, $M_G[V_{G_u}, V_G \setminus V_{G_u}]$ and $(M_G[V_G \setminus V_{G_u}, V_{G_u}])^T$. That means, for each $z \in X_{u_1}^+ \cup X_{u_2}^+$, there exists a row vector b_z such that $M_G[z, V_G \setminus V_{G_u}] = b_z \cdot M_G[X_u^+, V_G \setminus V_{G_u}]$. Similarly, for each $z' \in X_{u_1}^- \cup X_{u_2}^-$, there exists a row vector b'_z such that $M_G[V_G \setminus V_{G_u}, z] = b'_z \cdot M_G[V_G \setminus V_{G_u}, X_u^-]$. We let $t_u = t_{u_1} \otimes_{M_1, M_2, N_1, N_2, P_1, P_2} t_{u_2}$ where:

$$\begin{aligned} N_1 &:= \begin{pmatrix} b_{x_{i_1}} & \cdots & b_{x_{i_h}} \end{pmatrix}^T & P_1 &:= \begin{pmatrix} b_{y_{t_1}} & \cdots & b_{y_{t_h}} \end{pmatrix}^T \\ N_2 &:= \begin{pmatrix} b'_{x_{j_1}} & \cdots & b'_{x_{j_{h'}}} \end{pmatrix}^T & P_2 &:= \begin{pmatrix} b'_{y_{s_1}} & \cdots & b'_{y_{s_{k'}}} \end{pmatrix}^T \end{aligned}$$

From Claim 5.20 it remains to show that $\Gamma_H^+ \cdot N_1 \cdot M_G[X_u^+, V_G \setminus V_{G_u}] = M_G[V_{G_{u_1}}, V_G \setminus V_{G_u}]$ and $\Gamma_K^+ \cdot P_1 \cdot M_G[X_u^+, V_G \setminus V_{G_u}] = M_G[V_{G_{u_2}}, V_G \setminus V_{G_u}]$, and $M_G[V_G \setminus V_{G_u}, X_u^-] \cdot (\Gamma_H^- \cdot N_2)^T = M_G[V_G \setminus V_{G_u}, V_{G_{u_1}}]$ and $M_G[V_G \setminus V_{G_u}, X_u^-] \cdot (\Gamma_K^- \cdot P_2)^T = M_G[V_G \setminus V_{G_u}, V_{G_{u_2}}]$. But, this is a straightforward computation by the definitions of N_1 , N_2 , P_1 and P_2 , and X_u^+ and X_u^- . This concludes the proof of the lemma. \square

6 Conclusion

We extended the rank-width and some related results from the undirected graphs to the C -graphs. Presented results imply in particular that every MSOL-definable property can be checked in polynomial time on C -graphs, when C is finite. Every open question for the undirected case are of course still relevant for the C -graphs.

Recently, some authors investigated the clique-width of multigraphs [8] or weighted graphs [19]. These graphs can be seen as \mathbb{N} -graphs. It is straightforward to verify that the rank-width is not equivalent to the clique-width when C is infinite. It would be interesting to investigate the rank-width over an infinite field, and in particular its algorithmic aspects: the recognition of C -graphs of bounded rank-width, and the property checking on C -graphs of bounded rank-width.

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